UNIVERSITY OF NAIROBI
COLLEGE OF BIOLOGICAL AND PHYSICAL SCIENCES
School of Physical Sciences
Bachelor of Science in Meteorology

COURSE UNIT:
DYNAMIC METEOROLOGY II

(SMR 401)

Study Unit

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INTRODUCTION TO THE COURSE UNIT
Welcome to Dynamic Meteorology (SMR401). This course is a continuation of Dynamic Meteorology I (SMR401). If you studied SMR301 a long while ago, it may be advisable to review it once more before you embark on this course. As you worked through SMR301, you may have been introduced to several equations and may be wondering why this course appear to be mathematical. Well, as you may have already found out, there are many processes that take place in the atmosphere, dynamic meteorology will seek not only to explain how this comes about, but also to express the relationship between the forces involved in a mathematical form.

The next question you ask, why bother about mathematical expressions when the descriptive explanation suffice. The answer is three fold. First, the mathematical expressions. saves you the trouble to have to give long explanations and indeed some concepts cannot even be adequately explained in words. Secondly, the core business of a meteorologist is to forecast the future state of the weather and the most objective method of forecasting weather is by numerical weather prediction that is depedent on the mathematical formulation of atmospheric processes. I believe you would like to be an expert in numerical weather prediction. Thirdly, dynamic meteorology is key to understanding other courses in meteorology such as tropical meteorology, synoptic meteorology and climate modeling.

Having highlighted the importance of meteorology, let us now examine the scope of this course. The areas covered in this course unit in clued the following:

- Computational methods in dynamic meteorology
- Atmospheric kinematics
- Vorticity and divergence equations
- Circulation theorems
- Continuity and pressure tendency equations
- Generalized wave equations
- Atmospheric waves
- Introduction to numerical weather prediction (NWP)
Dynamic instabilities

<table>
<thead>
<tr>
<th>The course unit objectives</th>
</tr>
</thead>
<tbody>
<tr>
<td>At the end of this course you should be able to:</td>
</tr>
<tr>
<td>1. Perform computations in dynamic meteorology</td>
</tr>
<tr>
<td>2. Describe the various atmospheric kinematics</td>
</tr>
<tr>
<td>3. Derive the divergence and vorticity equation and discuss their applications</td>
</tr>
<tr>
<td>4. Discuss the circulation theorems and applications to various atmospheric motions.</td>
</tr>
<tr>
<td>5. Derive the continuity and pressure tendency equations and discuss their applications to atmospheric motions.</td>
</tr>
<tr>
<td>6. State how the generalized wave theory apply to the atmospheric oscillations</td>
</tr>
<tr>
<td>7. Use the perturbation theory to obtain the phase speed of various types of waves in the atmosphere</td>
</tr>
<tr>
<td>8. Describe the basic steps involved in developing the numerical weather prediction models</td>
</tr>
<tr>
<td>9. Discuss the dynamic instabilities</td>
</tr>
</tbody>
</table>
LECTURE 1

INTRODUCTION TO COMPUTATION METHODS IN DYNAMIC METEOROLOGY

1.0 Introduction to lecture 1

In this lecture you are going to learn about the methods that are used to estimate various atmospheric circulation parameter from the observed data. The skill acquired in this lecture will applied in the subsequent lectures.

1.1 Objectives of lecture 1

At the end of this lecture you should be able to.

1. Write down the forward, backward and centred differently scheme.
2. Compute the geostrophic wind using pressure gradient and geopotential gradients.
3. Compute the thermal wind between constant pressure levels and constant geopotential levels.
4. Compute divergence and vorticity over regular grid and curved motion.
5. Compute deformation type I and type II on regular grid.

1.2 Differencing Schemes

Let $f$ be a function dependent on $x$ and $y$

Thus $f = f(x, y)$

The partial difference with respect to $x$ is $\frac{\partial f}{\partial x}$ and the partial difference with respect to $y$ is $\frac{\partial f}{\partial y}$

If the function $f$ is known say $f = xy^2$

Then $\frac{\partial f}{\partial x} = y^2$ (1.3)

And $\frac{\partial f}{\partial y} = 2xy$. (1.4)
The values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ can be computed for any given point $(x, y)$.

However for many atmospheric circulation parameters their functions are unknown and their difference have to be estimated numerically.

### 1.2.1 Forward Differencing Scheme

The Taylor expansion of the function $f(x)$, where $x = x_o + \Delta x$

$$f(x) = f(x_o) + \frac{\partial f}{\partial x} \Delta x + \frac{\partial^2 f}{\partial x^2} \Delta x^2 + \frac{\partial^3 f}{\partial x^3} \Delta x^3 + \ldots$$

making $\frac{\partial f}{\partial x}$ the subject we obtain

$$\frac{\partial f}{\partial x} = \frac{f(x) - f(x_o)}{\Delta x} - \frac{\partial^2 f}{\partial x^2} \Delta x - \frac{\partial^3 f}{\partial x^3} \Delta x^2 - \ldots$$

If we retain only the term in the right hand side we have

$$\frac{\partial f}{\partial x} = \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x}$$

**Activity 1.1 Forward differencing Scheme**

1. Fill the dashes in the table 1.1 below.
Note that we cannot compute $\frac{\partial f}{\partial x}$ at $x=6$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$\frac{\partial f}{\partial x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>32</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>50</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>72</td>
<td>-</td>
</tr>
</tbody>
</table>

### 1.2.2 Backward Differencing Scheme

The Taylor expansion of the function $f(x)$ where $x = x_o - \Delta x$ is

$$f(x_o - \Delta x) = f(x_o) - \frac{\partial f}{\partial x} \Delta x + \frac{\partial^2 f}{\partial x^2} \Delta x^2 - \frac{\partial^3 f}{\partial x^3} \Delta x^3 + \ldots \ldots \quad (1.8)$$

Making $\frac{\partial f}{\partial x}$ the subject we have,

$$\frac{\partial f}{\partial x} = \frac{f(x_o) - f(x_o - \Delta x)}{\Delta x} + \frac{\partial^2 f}{\partial x^2} \Delta x - \frac{\partial^3 f}{\partial x^3} \Delta x^2 + \ldots \ldots \quad (1.9)$$

Dropping the other terms and retaining the first term we have

$$\frac{\partial f}{\partial x} = \frac{f(x_o) - f(x_o - \Delta x)}{\Delta x} \quad (1.10)$$

which is the backward differencing scheme.
Activity 1.2

Filling in the missing values. Notice that for the forward we cannot compute at \( \frac{\partial f}{\partial x} \left( x = 1 \right) = 6 \) while for the backward we cannot compute \( \frac{\partial f}{\partial x} \left( x = 1 \right) \).

What do you notice about the solution for the backward and forward schemes.

### Table 1.2

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>Backward ( \frac{\partial f}{\partial x} )</th>
<th>Forward ( \frac{\partial f}{\partial x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>10</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>10</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>52</td>
<td>22</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>80</td>
<td>-</td>
<td>34</td>
</tr>
<tr>
<td>6</td>
<td>114</td>
<td>-</td>
<td>34</td>
</tr>
</tbody>
</table>

1.2.3 Centred Differencing Schemes

The Taylor expansion of the functions \( f(x + \Delta x) \) and \( f(x - \Delta x) \) are

\[
f(x + \Delta x) = f(x) + \frac{\partial f}{\partial x} \Delta x + \frac{\partial^2 f}{\partial x^2} \Delta x^2 + \frac{\partial^3 f}{\partial x^3} \Delta x^3 + \frac{\partial^4 f}{\partial x^4} \Delta x^4 + \ldots
\]

(1.11)

\[
f(x - \Delta x) = f(x) - \frac{\partial f}{\partial x} \Delta x + \frac{\partial^2 f}{\partial x^2} \Delta x^2 - \frac{\partial^3 f}{\partial x^3} \Delta x^3 + \frac{\partial^4 f}{\partial x^4} \Delta x^4 + \ldots
\]

(1.12)

subtracting (1.11) from (1.12) we obtain

\[
f(x + \Delta x) - f(x - \Delta x) = 2 \frac{\partial f}{\partial x} \Delta x + \frac{\partial^3 f}{\partial x^3} \Delta x^3 + \frac{\partial^5 f}{\partial x^5} \Delta x^5
\]

(1.13)
making \( \frac{\partial f}{\partial x} \) the subject we obtain

\[
\frac{\partial f}{\partial x} = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} - \frac{\partial^3 f}{\partial x^2} \Delta x^2 - \frac{\partial^5 f}{\partial x^4} \Delta x^4 + \ldots \tag{1.14}
\]

Retaining only the first term on the right hand side we obtain

\[
\frac{\partial f}{\partial x} = \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} \tag{1.15}
\]

which is the finite differencing for the centred scheme.

Activity 1.3

Complete table 1.3 by filling in the missing values. Notice that for the centred differencing scheme, you cannot evaluate \( \frac{\partial f}{\partial x} \) at \( x_i \) and \( \frac{\partial f}{\partial x} \) at \( x_n \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( \frac{\partial f}{\partial x} ) centre</th>
<th>( \frac{\partial f}{\partial x} ) forward</th>
<th>( \frac{\partial f}{\partial x} ) backward</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>2</td>
<td>6</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>1.0</td>
<td>6</td>
<td>-</td>
<td>-</td>
<td>8</td>
</tr>
<tr>
<td>1.5</td>
<td>12</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2.0</td>
<td>20</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2.5</td>
<td>30</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3.0</td>
<td>42</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3.5</td>
<td>56</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4.0</td>
<td>72</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4.5</td>
<td>90</td>
<td>38</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5.0</td>
<td>110</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The advantage of the centred difference is that it has a smaller transaction error compared to the forward and backward. Therefore for most problems, unless specified you should use the centred differencing scheme.

### 1.3 Computation of Geostrophic Wind

The geostrophic wind equation in x, y, z, t coordinate is

\[
\vec{V}_g = f\hat{k} \times \frac{1}{\rho} \nabla P
\]  

(1.16)

\[
U_g = -f \frac{\partial \rho}{\rho \partial y}
\]  

(1.17)

\[
V_g = f \frac{\partial \rho}{\rho \partial x}
\]  

(1.18)

The magnitude of geostrophic wind is

\[
|\vec{V}_g| = f \frac{\partial \rho}{\rho \partial n}
\]  

(1.19)

where \( \partial n \) is the normal distance to the isobars.

The geostrophic wind equation is x, y, p, t coordinates

\[
\vec{V}_g = f\hat{k} \times \nabla \phi
\]  

(1.20)

\[
u_g = f \frac{\partial \phi}{\partial y}
\]  

(1.21)

\[
v_g = \frac{\partial \phi}{\partial x}
\]  

(1.22)

The magnitude of the geostrophic wind is x, y, p, t in

\[
|\vec{V}_g| = f \frac{\partial \phi}{\partial n}
\]  

(1.23)
Activity 1.4

1. Write the zonal and meridional component of the finite difference in both \( z \) and \( p \) vertical coordinates.
2. Write the equation for the magnitude of geostrophic wind in finite difference form.
3. What is the zonal and meridional wind component if the isobars are running from North west to South east at interval of 5 mb with the gradient directed toward North east and the distance between the isobars is \( 2.5 \times 10^5 \text{ m} \) and the density \( \rho = 1.27 \text{ kg m}^{-3} \) at latitude
   (i) 30°N, (ii) 60°N, (iii) 80°N

1.4 Computation of Thermal Wind

The thermal wind in \( x, y, z, t \) coordinates

\[
\vec{V}_g = \frac{\alpha}{f} \hat{k} \times \nabla P
\]

\[
= \frac{RT}{f} \hat{k} \times \nabla (\ln P)
\]

\[
\frac{\partial \vec{V}_g}{\partial z} = \frac{\partial}{\partial z} \left( \frac{RT}{f} \hat{k} \times \nabla (\ln P) \right)
\]

\[
= \frac{R}{f} \frac{\partial T}{\partial z} \hat{k} \times \nabla (\ln P) + \frac{RT}{f} \hat{k} \times \nabla \frac{\partial}{\partial z} \ln P
\]
\[
\frac{\partial \vec{V}_g}{\partial P} = \frac{1}{f} \hat{k} \times \nabla \frac{\partial \phi}{\partial P} \quad (1.34)
\]

\[
\frac{\partial \vec{V}_g}{\partial P} = -\frac{R}{fP} \hat{k} \times \nabla T \quad (1.35)
\]

\[
\Delta \vec{V}_g = \int_{b_{\phi}}^{a_{\phi}} \left( \frac{R}{f} \hat{k} \times \nabla T \right) \partial P \quad (1.36)
\]

In the horizontal temperature gradient is constant with the layer then.

\[
\vec{V}_T = -\left( \frac{R}{f} \hat{k} \times \nabla T \right) \ln \frac{P_2}{P_1} \quad (1.38)
\]
\[ u_r = \frac{R}{f} \frac{\partial T}{\partial y} \ln \frac{P_2}{P_1} \quad \text{where} \quad u_r = u(P_2) - u(P_1) \quad (1.39) \]

\[ v_r = -\frac{R}{f} \frac{\partial T}{\partial x} \ln \frac{P_2}{P_1} \quad \text{and} \quad v_r = v(P_2) - v(P_1) \quad (1.40) \]

Activity 1.5

1. Determine the geostrophic wind at this level 2 km, if the geostrophic wind at 1 km is directed from south to north and its speed is 8.00 ms\(^{-1}\) and if the gradient of the mean temperature for the layer is directed from north to south, and its magnitude is 2.5\(^\circ\)C/100 km. The latitude is 60\(^\circ\)S.

2. Find the thermal wind in a layer 2 km thick if the horizontal temperature gradient is directed south east and is 2.5\(^\circ\)C/120 km, and the average temperature of the layer is \(\bar{T} = 275^\circ k\). The latitude is 50\(^\circ\)N.

3. Determine the geostrophic wind at 600 mb level, if the geostrophic wind at 850 mb is from north east and equal to 5 ms\(^{-1}\), and if the gradient of the mean temperature for the layer is directed south east and its magnitude is 4.5\(^\circ\)/150 km. The latitude is 30\(^\circ\)N.

1.5 Computation of Divergence

The divergence in three dimension is x, y, z, t coordinate.

\[ \nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \]

The horizontal divergence is the most important quantity.
\[ D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \]

In natural coordinate the divergence is given by

\[ D = \frac{\partial v}{\partial s} + V \frac{\partial \beta}{\partial n} \]

Where \( s \) is a natural coordinate along the positive direction of the streamline, \( n \) is along the normal to the streamline directed to the left of the \( s \) direction, \( V \) is the wind speed, and \( \beta \) is an angle between the tangent to the streamline and an arbitrary fixed direction.

**Activity 1.6**

1. Compute the divergence at the points (I-1, J), (I-1 J-1), (I, J), (I, J + 1), (I, J-1), (I + 1, J + 1), (I +1, J). Given the \( u \) and \( v \) at the points (I-1, J-1), (I-1, J), (I-1, J+1), (I, J), (I, J+1), (I, J-1), (I+1, J+1), (I+1,J), (I+1, J-1) are (12, 4), (-5, 6), (8, -5), (-6, 2), (10, 12), (12, 6), -4, -2), -6, 3) and \( \Delta x=\Delta y=200\text{km} \).

2. Determine the divergence of the wind field in a field of parallel streamlines, if the wind speed decreases by 4m/sec for each 500 km along the streamline.

3. Determine the divergence of the wind fields if the streamlines, being on the average 400 km apart, converge with an angle of 15°, the wind speed is constant and equal to 10m/sec.

**1.6 Computation of Vorticity**

Vorticity is a measure of the strength of rotation about a given axis.

\[ i - \text{axis} \quad \zeta_i = i.\nabla \times \vec{V} \]

The vorticity about \( j - \text{axis} \quad \zeta_j = j.\nabla \times \vec{V} \)

\[ k - \text{axis} \quad \zeta_k = k.\nabla \times \vec{V} \]

Where \( \vec{V} = u\hat{i} + v\hat{j} + w\hat{k} \)
$$\zeta_i = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$
$$\zeta_j = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$
$$\zeta_k = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The vertical vorticity is the most important and unless otherwise stated, vorticity (or relative vorticity) refer to the vertical vorticity.

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\zeta = V \kappa_s - \frac{\partial v}{\partial n}$$

where $\kappa_s$ is the curvature. Absolute vorticity $\eta$ is determined by the formulae

$$\eta = \zeta + f$$

where $f$ is the coriolis parameter $f = 2\Omega \sin \varphi$

Activity 1.7

1. Determine the vorticity of the wind field in a circular streamline of cyclonic curvature when the radius is 640 km and the wind speed at this and adjacent streamlines is equal to 12 m$^{-1}$

2. Determine the vorticity of the wind field on a circular streamline of anticyclonic curvature when the radius is 1500 km, the wind speed at this streamline is 9 m$^{-1}$ and the speed decreases linearly towards the centre of the anticyclone.

3. Determine the vorticity of the wind field in a flow with parallel streamlines, if the wind speed changes across the flow by 4 m/sec for each 500 km.
1.7 Computation of Deformation

Deformation of type I is defined as

\[ F_1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \]

and deformation type II by

\[ F_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \]

Activity 1.8

Compute the deformation type I and type II at the point (I, J). Given the wind \( u \), and \( v \) components at \((I + 1, J), (I, J), (I, J + 1), (I-1, J)\) and \((I, J-1)\) to be \((12, 2), (8, 9), (6, 6), (4, 5)\) and \((2, 4)\) and \( \Delta x = \Delta y = 150km \).

1.8 Computation of the Gradient Wind

The gradient wind is a balanced flow along a stationery circular isobar in the absence of friction. It is directed along the isobar in such a way, that the areas of lower pressure is to the left in the northern hemisphere and to the right in the southern hemisphere.

The gradient wind speed \( V_{gr} \) in a cyclone is related to the geostrophic wind speed \( V_g \), the coriolis parameter \( f \), and the radius of the curvature, \( r \) by the expression.

\[ V_{gr} = \frac{fr}{2} \left( -1 + \sqrt{1 + \frac{4V_g}{fr}} \right) \]

For an anticyclone we get
\[ V_g = \frac{fr}{2} \left( 1 - \sqrt{1 - \frac{4V_g}{fr}} \right) \]

For small magnitude of \( \frac{V_g}{fr} \) we approximate the formula.

\[ V_{gr} = V_g \left( 1 \pm \frac{V_g}{fr} \right) \]

**SUMMARY**

1. Given a differential equation \( \frac{\partial f}{\partial x} = f(x) \) the finite scheme form evaluated at \( x_o \) are
   (i) forward \( \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x} = F(x_o) \)
   (ii) Backward \( \frac{f(x_o) - f(x_o - \Delta x)}{\Delta x} = F(x_o) \)
   (iii) Centred \( \frac{f(x_o + \Delta x) - f(x_o - \Delta x)}{2\Delta x} = F(x_o) \)
   (iv) Trapezoidal \( \frac{f(x_o + \Delta x) - f(x_o)}{\Delta x} = \frac{1}{2} \left( F(x_o) + F(x_o - \Delta x) \right) \)

2. The centred scheme is preferred in most problems

3. Applying the centred scheme to obtain the pressure gradient at \((0,0)\), we get

\[
\nabla P = \frac{\partial P}{\partial x} i + \frac{\partial P}{\partial y} j
\]

\[
= \left( \frac{P(x_o + \Delta x) - P(x_o - \Delta x)}{2\Delta x} \right)_i + \left( \frac{P(y_o + \Delta y) - P(y_o - \Delta y)}{2\Delta y} \right)_j
\]

LECTURE 2

ATMOSPHERIC KINEMATICS

2.0 Introduction to Lecture 2

In this lecture, you will learn about the atmospheric kinematics, kinematics in the study of motions without reference to the forces causing them.

2.1 Objectives of Lecture 2

In the end of this lecture you should be able to

1. Derive the linear relationship between the horizontal wind and divergence, vorticity and deformation type I and II.

2. Derive the equations of streamlines and trajectory.

3. Demonstrate that under rotation of coordinates divergence, vorticity and sum of square of deformation type I and II are invariant.

4. Use Helmholtz theorem to resolve the horizontal wind into rotational and instational component.

5. Describe how the streamfunction and velocity potential are obtained from the observed wind field.

2.2 Linear Relationship Between Horizontal Wind and Divergence, Vorticity, Deformation Type I and II

When you look at the streamlines of the wind find, especially over a large domain, you will notice various patterns. For example you will see anti-cyclonic circulation, cyclonic circulation, a col and so on. In this section we show these patterns represent divergence, vorticity, deformation Type I and II

Consider two dimensional wind field.

\[ u = u(x, y) \quad \text{and} \quad v = v(x, y) \]

Define the origin \( x_0, y_0 \) and the corresponding velocity component at this point as \( u_0 \) and \( v_0 \).
We let \( x = x_o + \delta x \) and \( y = y_o + \delta y \) where \( \delta x \) and \( \delta y \) are small enough such that the point \((x, y)\) is in the neighbourhood of \((x_o, y_o)\). We can then use Taylor’s expansion theorem.

\[
\begin{align*}
  u &= u_o + \left( \frac{\partial u}{\partial x} \right)_o \delta x + \left( \frac{\partial u}{\partial y} \right)_o \delta y \quad \text{.......................... (2.1)}
  \\
v &= v_o + \left( \frac{\partial v}{\partial x} \right)_o \delta x + \left( \frac{\partial v}{\partial y} \right)_o \delta y \quad \text{.......................... (2.2)}
\end{align*}
\]

We can rewrite (2.1) and (2.2) in the form

\[
\begin{align*}
  u &= u_o + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)_o \delta x + \frac{1}{2} \left( \frac{\partial u}{\partial y} \right)_o \delta y + \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)_o \delta x + \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)_o \delta y - \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)_o \delta y \quad \text{.......................... (2.3)}
  \\
v &= v_o + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)_o \delta x + \frac{1}{2} \left( \frac{\partial v}{\partial y} \right)_o \delta y + \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)_o \delta x + \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_o \delta y - \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)_o \delta y \quad \text{.......................... (2.4)}
\end{align*}
\]

Recall from lecture one that

\[
\begin{align*}
  D &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad \text{.......................... (2.5)}
  \\
  \zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{.......................... (2.6)}
  \\
  F_1 &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \quad \text{.......................... (2.7)}
  \\
  F_2 &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \text{.......................... (2.8)}
\end{align*}
\]

Substituting (2.5)to (2.8) into (2.3) and (2.4) we obtain

\[
\begin{align*}
  u &= u_o + \frac{1}{2} D_o \delta x - \frac{1}{2} \zeta_o \delta y + \frac{1}{2} F_{1o} \delta x + \frac{1}{2} F_{2o} \delta y \quad \text{..........................(2.9)}
  \\
v &= v_o + \frac{1}{2} D_o \delta x + \frac{1}{2} \zeta_o \delta x - \frac{1}{2} F_{1o} \delta y + \frac{1}{2} F_{2o} \delta x \quad \text{..........................(2.10)}
\end{align*}
\]
Equations (2.9) and (2.10) shows that the changes in the wind pattern as it moves from point \((x_o, y_o)\) to \((x, y)\) is influenced by the forcings, \(D, \zeta, F_1\) and \(F_2\).

Activity 2.1

1. Given that
   \[ u = u_o x - u_o \sin ky \]
   \[ v = v_o y + v_o \cos kx \]
   where \(u_o=10\text{ms}^{-1}, v_o=2\text{ms}^{-1}, k=2\times10^{-3}, x_o=0\) and \(y_o=0\). Compute \(D_o, \zeta_o, F_{1o}\) and \(F_{2o}\).

2. Find the wind speed and direction at a point \((x=60\text{km}, y=40\text{km})\) given that
   \[ u_o=8\text{ms}^{-1}, v_o=6\text{ms}^{-1}, D_o=-2\times10^{-5}\text{s}^{-1}, \zeta_o=8\times10^{-5}\text{s}^{-1}, F_{1o}=-6\times10^{-5}\text{s}^{-1}\] and \(F_{2o}=4\times10^{-5}\text{s}^{-1}\).

### 2.3. Equation of Streamline and Trajectory

#### 2.3.1 Equation of Streamline

A streamline is a line such that everywhere on it, the instantaneous wind vector is tangential to it. Writing the \(u\) and \(v\) as the rate of change of

\[
\frac{dx}{dt} = u \quad \text{(2.11)}
\]

\[
\frac{dy}{dt} = v \quad \text{(2.12)}
\]

Dividing (2.12) by (2.11) we have

\[
\frac{dy}{dx} = \frac{v}{u} \quad \text{................. (2.13)}
\]

If \(u\) and \(v\) are functions of \(x\) and \(y\), equation (2.13) can be integrated to obtain the equation of the streamlines. For example

\[
\vec{V} = \frac{5}{y} i + kj \quad \text{................. (2.14)}
\]

Using equation (2.13) we have

\[
\frac{dy}{dx} = \frac{ky}{5} \quad \text{......... (2.15)}
\]

\[
\frac{1}{y} \frac{dy}{dx} = \frac{k}{5} \quad \text{......... (2.16)}
\]

Integrating we obtain the equation of the streamline.
\[ \ln(y) = \frac{1}{3} x + c \]
\[ y = e^{\frac{1}{3} x + c} \] \hspace{1cm} \text{……… (2.17)}

Where \( c \) is the constant of integration

Let us now look at some simple flow patterns

**Case 1: Purely Divergent Flow**

Consider a case where \( D = \text{constant} \), \( \zeta = F_1 = F_2 = 0 \)

Recall equation (9) and (10)

\[
\begin{align*}
  u &= u_o + \frac{1}{2} D \delta x - \frac{1}{2} \beta \delta y + \frac{1}{2} F_1 \delta x + \frac{1}{2} F_2 \delta y \\
  v &= v_o + \frac{1}{2} D \delta y + \frac{1}{2} \beta \delta x + \frac{1}{2} F_1 \delta y + \frac{1}{2} F_2 \delta y
\end{align*}
\]

Let \( x_o = y_o = 0 \)

And \( u_o = u = 0 \)

Therefore \( x = \delta x \) and \( y = \delta y \)

Hence \( u = \frac{1}{2} D x \) \hspace{1cm} \text{…………….. (2.16)}

\[ v = \frac{1}{2} D y \] \hspace{1cm} \text{…………….. (2.17)}

From (13)

\[
\begin{align*}
  \frac{dy}{dx} &= \frac{1}{2} \frac{D y}{D x} \\
  \frac{dy}{dx} &= \frac{y}{x}
\end{align*}
\]

\[
\begin{align*}
  \frac{1}{y} \frac{dy}{dx} &= \frac{1}{x} \\
  d(ln y) &= d(ln x)
\end{align*}
\]

Integrating we obtain

\( ln y = ln x + c' \)
Let \( Inc = e' \)

Then \( y = cx \) .......................... (2.18)

Equation (2.18) represents equation of straight lines through the origin (0,0). The arrows on the streamlines point in the direction of the wind flow determined by the vector \( \vec{V} \).

For the above case \( \vec{V} = \frac{1}{2} Dx\hat{x} + \frac{1}{2} Dy\hat{y} \) If D is positive the flow will from the centre (0,0). And if D is negative the flow will be inward toward (0,0).

**Case 2: Purely Rotational Flow**

Consider a case where \( \zeta = \) constant and \( D = F_1 = F_2 = 0 \)

Let again \( x_o = y_o = 0 \) and \( u_o = u_o = 0 \)

Then

\[
\begin{align*}
  u &= -\frac{1}{2} \zeta y \\
  v &= \frac{1}{2} \zeta x \\
  \frac{dy}{dx} &= \frac{1}{2} \zeta x - \frac{1}{2} \zeta y
\end{align*}
\]

\( ydy = -xdx \)

Integrating we obtain

\( y^2 = -x^2 + C' \)

Let \( C' = C^2 \)

\( y^2 + x^2 = C^2 \)

The streamlines are concentric circles with the centre at \( x = 0, y = 0 \)

The circulation will be anticlockwise when \( \zeta \) is positive and clockwise when \( \zeta \) is negative.

**Case 3: Circulation Having only Deformation Type I**

\( F_1 = \) constant and \( D = \zeta = F_2 = 0 \)
Let $x_o = y_o = 0$ and $U_o = V_o = 0$ 

$$u = \frac{1}{2} F_1 x$$

$$u = -\frac{1}{2} F_1 y$$

$$\frac{dy}{dx} = -\frac{1}{2} \frac{F_1 y}{F_1 x}$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$d \ln y = d \ln x$$

Integrating we obtain

$$\ln y = - \ln x + C'$$

Let $\ln C = C'$

$$\ln y = - \ln x + \ln C$$

$$\ln y = \ln \frac{C}{x}$$

$$y = \frac{C}{x}$$

The streamlines are hyperbolic with the $x$ and $y$-axis as asymptotes.

---

**Case 4: Wind flow having only Deformation Type II**

$F_2$ = constant and $D = \zeta = F_1 = 0$

Let $x_o = y_o = 0$ and $u_o = v_o = 0$
\[ u = \frac{1}{2} F_y y \]

\[ V = \frac{1}{2} F_x x \]

\[ \frac{dy}{dx} = \frac{\frac{1}{2} F_x}{\frac{1}{2} F_y} \]

\[ ydy = xdx \]

Integrating we obtain

\[ y^2 - x^2 = C^2 \]

\[ y = \sqrt{C^2 + x^2} \]

The streamlines are hyperbolic with lines \( y = x \) and \( y = -x \) as asymptotes

**Activity 2.2**

1. Draw schematic diagram for the wind vectors given by
   a) \( \vec{V} = -x\hat{i} + o\hat{j} \)
   b) \( \vec{V} = x\hat{i} + o\hat{j} \)
   c) \( \vec{V} = o\hat{i} - x\hat{j} \)
   d) \( \vec{V} = o\hat{i} + x\hat{j} \)
   e) \( \vec{V} = x\hat{i} + y\hat{j} \)
   f) \( \vec{V} = -x\hat{i} + y\hat{j} \)
   g) \( \vec{V} = -x\hat{i} - y\hat{j} \)
   h) \( \vec{V} = y\hat{i} + x\hat{j} \)
   i) \( \vec{V} = -y\hat{i} + x\hat{j} \)
   j) \( \vec{V} = y\hat{i} - x\hat{j} \)
   k) \( \vec{V} = -y\hat{i} - x\hat{j} \)

2. Derive the equations of the streamline, corresponding to each of the wind vector given in 1 and draw the streamline.
2.3.2. Equation of Trajectory

A trajectory is the actual path followed by an air parcel. The trajectory is defined by the position vector.

\[ \vec{R} = x \hat{i} + y \hat{j} \]

and is obtained by integrating the wind components with respect to time.

\[ x = \int_{t_0}^{t} U \, dt \quad \text{and} \quad x = \int_{t_0}^{t} V \, dt \]

Example

\[ \vec{V} = x \hat{i} + y \hat{j} \]

ie \[ \frac{dx}{dt} = x \]

and \[ \frac{dy}{dt} = y \]

\[ \frac{1}{x} \, dx = \, dt \]

\[ d(1 - x) = \, dt \]

\[ \int_{x_0}^{x} \, d(Inx) = \int_{t_0}^{t} \, dt \]

\[ In\left(\frac{x}{x_0}\right) = t - t_0 \]

\[ x = x_0 \, e^{t-t_0} \]

Similarly

\[ y = y_0 \, e^{t-t_0} \]

Thus

\[ \vec{R} = x_0 \, e^{t-t_0} \hat{i} + y_0 \, e^{t-t_0} \hat{j} \]
Let us consider the case where

\[ u = -\frac{1}{2} \zeta y \]

\[ u = \frac{1}{2} \zeta x \]

Thus

\[ \frac{dx}{dt} = -\frac{1}{2} \zeta y \]

\[ \frac{dy}{dt} = \frac{1}{2} \zeta x \]

Differentiating the first equation w.r.t. time, \( t \)

\[ \frac{d^2x}{dt^2} = -\frac{1}{2} \zeta \frac{dy}{dt} \]

Substituting with the second equation

\[ \frac{d^2x}{dt^2} = -\frac{1}{4} \zeta^2 x \]

\[ \frac{d^2x}{dt^2} + \frac{1}{4} \zeta^2 x = 0 \]

The solution is

\[ x = A \cos \left( \frac{1}{2} \zeta t \right) + B \sin \left( \frac{1}{2} \zeta t \right) \]

Differentiating the second equation w.r.t. time, \( t \) we obtain

\[ \frac{d^2y}{dt^2} = \frac{1}{2} \xi \frac{dx}{dt} \]

Substituting with the first equation we obtain
\[ \frac{d^2 y}{dt^2} = -\frac{1}{4}\zeta^2 y \]

\[ \frac{d^2 y}{dt^2} + \frac{1}{4}\zeta^2 y = 0 \]

The solution is

\[ y = C \cos \frac{1}{2}\zeta t + D \sin \frac{1}{2}\zeta t \]

To solve for A, B, C and D we consider the initial conditions.

\[ x = A \cos \frac{1}{2}\zeta t + B \sin \frac{1}{2}\zeta t \]

\[ y = C \cos \frac{1}{2}\zeta t + D \sin \frac{1}{2}\zeta t \]

at \( t = 0 \)

\( x_o = A \) and \( C = y_o \)

Differentiating w.r.t time we obtain,

\[ u = -\frac{1}{2}\zeta A \sin \frac{1}{2}\zeta t + \frac{1}{2}\zeta B \cos \frac{1}{2}\zeta t \]

\[ v = -\frac{1}{2}\zeta C \sin \frac{1}{2}\zeta t + \frac{1}{2}\zeta D \cos \frac{1}{2}\zeta t \]

ie

\[-\frac{1}{2}\zeta y = -\frac{1}{2}\zeta A \sin \frac{1}{2}\zeta t + \frac{1}{2}\zeta B \cos \frac{1}{2}\zeta t \]

Thus at \( t = 0 \)

\[ B = -y_o \]

And

\[ \frac{1}{2}\zeta x = -\frac{1}{2}\zeta C \sin \frac{1}{2}\zeta t + \frac{1}{2}\zeta D \cos \frac{1}{2}\zeta t \]
at $t = 0$

$$D = x_o$$

The equation of the trajectory is

$$\vec{R} = x \hat{i} + y \hat{j}$$

where

$$x = x_o \cos \frac{1}{2} \zeta t - y_o \sin \frac{1}{2} \zeta t$$

$$y = y_o \cos \frac{1}{2} \zeta t + x_o \sin \frac{1}{2} \zeta t$$

Squaring

$$x^2 = x_o^2 \cos^2 \frac{1}{2} \zeta t - 2x_o y_o \cos \frac{1}{2} \zeta t \sin \frac{1}{2} \zeta t + y_o^2 \sin^2 \frac{1}{2} \zeta t$$

$$y^2 = y_o^2 \cos^2 \frac{1}{2} \zeta t + 2x_o y_o \cos \frac{1}{2} \zeta t \sin \frac{1}{2} \zeta t + x_o^2 \sin^2 \frac{1}{2} \zeta t$$

adding

$$x^2 + y^2 = \cos^2 \frac{1}{2} \zeta t (x_o^2 + y_o^2) + (y_o^2 + x_o^2) \sin^2 \frac{1}{2} \zeta t$$

$$= (x_o^2 + y_o^2) \left( \cos^2 \frac{1}{2} \zeta t + \sin^2 \frac{1}{2} \zeta t \right)$$

$$x^2 + y^2 = x_o^2 + y_o^2$$

This show that the trajectory are the same as the streamline.

**Activity 2.3**

Find the equation of the streamline and trajectory for the following

$$u = U$$

$$v = V \cos k(x - ct)$$

Express the phase velocity in terms of the streamline amplitude and trajectory amplitude
Streaklines

A streakline is a line joining the air parcels that have passed through a given point.

2.4 Invariant of Divergence, Vorticity and Deformation under Rotation of Coordinates

Consider a fixed point in space $P$, it will have the coordinates $(x, y)$ in the unrotated axis. When the axis are rotated through an angle $\theta$, the coordinates of $P$ will be $(x, y)$.

We express $x_1$ and $y_1$ in terms of $x$ and $y$ thus

\[
x_1 = x_1 (x, y)
\]

\[
y_1 = y_1 (x, y)
\]

Using geometry we find that

\[
x_1 = OE + EB
\]

\[
OE = x \cos \theta
\]

\[
EB = DP = y \sin \theta
\]

\[
x_1 = x \cos \theta + y \sin \theta
\]

\[
y_1 = AD - AE
\]

\[
AO = y \cos \theta
\]

\[
AE = x \sin \theta
\]

\[
y_1 = y \cos \theta - x \sin \theta
\]

We also see that

\[
x = OA' - AA'
\]

\[
OA' = x_1 \cos \theta
\]

\[
AA' = PC
\]
\[ y = y_1 \sin \theta \]
\[ x = x_1 \cos \theta - y_1 \sin \theta \]
\[ y = A'B + BC \]
\[ A'B = x_1 \sin \theta \]
\[ BC = y_1 \cos \theta \]
\[ y = x_1 \sin \theta + y_1 \cos \theta \]

**Activity**

Given

\[ x = x_1 \cos \beta - y_1 \sin \beta \]
\[ y = x_1 \sin \beta + y_1 \cos \beta \]

Express \( x_1 \) and \( y_1 \) in terms of \( x \) and \( y \)

We have the equations

\[ x_1 = x \cos \theta + y \sin \theta \]
\[ y_1 = -x \sin \theta + y \cos \theta \]

and

\[ x = x_1 \cos \theta - y_1 \sin \theta \]
\[ y = -x_1 \sin \theta + y_1 \cos \theta \]

Differentiating equation ( ) and ( ) w.r.t. time we obtain

\[ U_1 = U \cos \theta + V \sin \theta \]
\[ V_1 = -U \sin \theta + V \cos \theta \]
To show that divergence is invariant under rotation of axis, we prove that

$$\frac{\partial u}{\partial x_i} + \frac{\partial v}{\partial y_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

and

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial x_i} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial x_i}$$

$$\frac{\partial v}{\partial y_i} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial y_i} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial y_i}$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} \cos \theta + \frac{\partial v}{\partial y} \sin \theta$$

$$\frac{\partial y}{\partial x} = \cos \theta$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} \cos \theta + \frac{\partial v}{\partial y} \sin \theta$$

$$\frac{\partial y}{\partial y} = \sin \theta$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} \cos \theta - \frac{\partial u}{\partial x} \sin \theta$$

$$\frac{\partial x}{\partial y_i} = -\sin \theta$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} \cos \theta - \frac{\partial u}{\partial y} \sin \theta$$

$$\frac{\partial y}{\partial y_i} = \cos \theta$$

Putting the terms together, we obtain
\[
\frac{\partial u_i}{\partial x_i} + \frac{\partial v_i}{\partial y_i} = \left( \frac{\partial u}{\partial x} \cos \theta + \frac{\partial v}{\partial x} \sin \theta \right) \cos \theta + \left( \frac{\partial u}{\partial y} \cos \theta + \frac{\partial v}{\partial y} \sin \theta \right) \sin \theta \\
+ \left( \frac{\partial v}{\partial x} \cos \theta - \frac{\partial u}{\partial x} \sin \theta \right) (-\sin \theta) + \left( \frac{\partial v}{\partial y} \cos \theta - \frac{\partial u}{\partial y} \sin \theta \right) \cos \theta
\]

Simplifying the right hand results to

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
\]

Thus we have demonstrated that divergence is variant under rotation of axis.

**Activity 2.4**

1. Show that vorticity is invariant under rotation of coordinates.

   Thus

   \[
   \frac{\partial v_i}{\partial x_i} - \frac{\partial u_i}{\partial y_i} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
   \]

2. Show that the sum of square of deformation type I and type II is invariant under rotation of coordinates.

   \[
   F_{11}^2 + F_{12}^2 = F_1^2 + F_2^2
   \]

**2.5 Helmholtz Theorem of Resolution of horizontal wind into rotational and irrotational component**

Helmholtz theorem states the wind component can be resolved into rotation (non divergent) and irritation (divergent) component.

\[
\vec{V} = \vec{V}_{ND} + \vec{V}_D
\]

Or

\[
\vec{V} = \vec{V}_R + \vec{V}_{IR}
\]
This means that
\[ K \nabla \times \vec{V}_D = 0 \]

and
\[ \nabla \cdot \vec{V}_{ND} = 0 \]

From vector analysis if \( \vec{V}_D \) is a non nul vector, then it should be the ascendant (Grad) of some scalar function in this case we call velocity potential.

Thus
\[ \vec{V}_D = \nabla \chi \]

where \( \chi \) is velocity potential.

The rotational component can be defined such that the dive of that vector is zero.

Thus
\[ \vec{V}_{ND} = \hat{k} \times \nabla \psi \]

The function \( \psi \) is called the stream function.

The components of the divergent components are
\[ U_x = \frac{\partial \chi}{\partial x} \quad \text{and} \quad V_z = \frac{\partial \chi}{\partial y} \]

While the components of the non-divergent components are
\[ u_\psi = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v_\psi = \frac{\partial \psi}{\partial x} \]

**Activity 2.5**

1. We define divergence \( D \) as
\[ D = \nabla \cdot \vec{V} \]
where \( \vec{V} = \vec{V}_\chi + \vec{V}_\psi \)

Show \( D = \nabla^2 \chi \)

2. Similarly we define \( \zeta \) as

\[ \zeta = k \nabla \times \vec{V} \]

show that \( \zeta = \nabla^2 \psi \)

3. Write deformation type I, \( F_1 \) and deformation type II \( F_2 \) in terms of \( \psi \) and \( \chi \)

Velocity potential \( \chi \), and the stream function \( \Psi \) are very useful fields in analysis of the wind flow pattern. When the velocity potential field are analysed, the streamlines of the divergent wind will be perpendicular to the isolines of the velocity potential and directed towards regions of high values. In other words, low value of velocity potential indicate divergence which high values of velocity potential indicate divergence.

On the other hand when the stream function have been analyzed, the streamline of the rotational wind will coincide with the stream function, with flow such that the high values of the stream function are to the right of motion.

2.6 Computation of Velocity Potential and Stream function from the Observed Wind field.

The observed wind field will contain both the divergent and non-divergent wind components.

\[ \vec{V} = \vec{V}_{ND} + \vec{V}_D \]

We showed that

\[ \nabla^2 \chi = D \]
Where D is divergence computed from the observed wind.

Similarly

\[ \nabla^2 \psi = \zeta \]

where \( \zeta \) is computed from the observed wind field.

Usually the observed wind field consist of wind speed and direction. The steps of computing velocity potential and stream function are:

1. Resolve the observed wind into zonal, \( u \) and meridional, \( v \) components.

2. Compute divergence, \( D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \) and vorticity \( \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \).

3. Solve the Laplacian equations

\[ \nabla^2 \chi = D \]

and \( \nabla^2 \psi = \zeta \)

4. Finally compute the rotational and divergence wind component by

\[ u_{ND} = -\frac{\partial \psi}{\partial y} \; \text{and} \; v_{ND} = \frac{\partial \psi}{\partial x} \]

and

\[ u_D = \frac{\partial \chi}{\partial x} \; \text{and} \; v_D = \frac{\partial \chi}{\partial y} \]

Note:

Although we can have a program that compute \( \psi \) and \( \chi \) simultaneously, sometimes we may be interested in one of the field.

To obtain \( \psi \) or \( \chi \) we use a method referred to as relaxation method.

2.7 Relaxation Method

Consider the equation
\[ \nabla^2 \phi = F \]

The left hand side provides the Laplace of \( \phi \) and the term on the right hand side is called the forcing.

Relaxation method is a simple numerical method of finding values of a function \( \phi \) at internal grid points when

a) We are given values of \( F \) inner grid points and we
b) Are given suitable values of \( \phi \) at the boundary.

For simplicity we let

\[ \Delta x = \Delta y = d \]

be the grid distance

\( M \) the number of grid spacing in the x-direction
\( \) And \( N \) the number of grid spacing in the y-direction

**Steps of relaxation method**

1. Replace \( \nabla^2 \phi = F \) with finite difference equation, for example at \( I, J \).

\[
\nabla^2 \phi \bigg|_{I,J} = \frac{\partial^2 \phi}{\partial x^2} \bigg|_{I,J} + \frac{\partial^2 \phi}{\partial y^2} \bigg|_{I,J} \\
\frac{\partial^2 \phi}{\partial x^2} \bigg|_{I,J} = \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right)_{I,J} = \left( \frac{\phi_{I+1,J} - \phi_{I,J}}{d} - \left( \frac{\phi_{I,J} - \phi_{I-1,J}}{d} \right) \right) / d \\
\frac{\partial^2 \phi}{\partial y^2} \bigg|_{I,J} = \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial y} \right)_{I,J} = \frac{\phi_{I+1,J} + \phi_{I-1,J} - 2\phi_{I,J}}{d^2} 
\]
\[
\frac{\partial^2 \phi}{\partial y^2}_{i,j} = \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial y} \right)_{i,j} = \left( \frac{\phi_{i,j+1} - \phi_{i,j} - \phi_{i,j} - \phi_{i,j-1}}{d} - \frac{\phi_{i,j+1} - \phi_{i,j-1}}{d} \right) / d
\]

\[
\phi_{i,j+1} + \phi_{i,j-1} - \partial \phi_{i,j}
\]

\[
\frac{\phi_{i,j+1} + \phi_{i,j-1} - \partial \phi_{i,j}}{d^2}
\]

\[
\nabla^2 \phi = \left( \phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} \right) / d^2 = 0
\]

Thus \( \nabla^2 \phi = F \) is replaced by the finite difference

\[
\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} - d^2 F_{i,j} = 0
\]

These are \((M - 1) (N - 1)\) internal grid points. Notice therefore that \( \phi \) cannot be computed at the boundary. Suitable values of \( \phi \) must be defined at the boundary.

There are \((M - 1) (N - 1)\) equations with \((M - 1) (N - 1)\) unknown, therefore the system is complete.

**Why do we need the boundary values?**

We need the boundary values so that we can be able to write the difference equation at the internal grid points neighbouring the boundary.

2. Prescribe guess values for \( \phi \) at all internal grid points. Once this is done, the right hand side of the equation will not necessarily be zero. There will be residue, say \( R \).

\[
\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} - d^2 F_{i,j} - R_{i,j}
\]

The method used is iteration method which gives a temporal increment to \( \phi_{i,j} \) so that the residual at the grid point is temporarily increased, so that the residual at that grid point is temporarily reduced to zero.

\[
\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} - d^2 F_{i,j} = R_{i,j}
\]

\[
\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\left( \phi_{i,j} + \frac{1}{4} R_{i,j} \right) - d^2 F_{i,j} = 0
\]

where \( \phi_{i,j} + \frac{1}{4} R_{i,j} = \phi_{i,j}^{\prime} \) (new value)

Repeat the operation at the next grid point, say \( I + 1, J \)
\[ \phi'_{I+1,J} = \phi_{I+1,J} + \frac{1}{4} R_{I+1,J} \]

The process is continued until the last value is modified i.e.

\[ \phi'_{M-1,N-1} = \phi_{M-1,N-1} + \frac{1}{4} R_{M-1,N-1} \]

This will constitute a complete scan. By the time a complete scan is done, the residue at the respective points is not necessarily zero. However, as the number of scan increases, the magnitude of the residue decreases at the grid point. In general, the process converges.

In practice it is not possible to liquidate the residue to zero. So you set residue limit \( R_L \). The scanning is stopped when for all grid points \( R_{I,J} < R_L \).

---

**SUMMARY**

1. A streamline is a line that is everywhere tangential to the wind direction.
2. Given a wind field \( \vec{V} = g(x, y)i + h(x, y)j \), then the equation of the streamline is obtained by integrating the differential equation \( \frac{dy}{dx} = \frac{h}{g} \).
3. A trajectory is the actual path followed by the parcel of air. For the wind field in (2), the trajectory would be obtained as follows:
   \[ \vec{r} = xi + yj \quad \text{where} \quad x = \int_0^t g dt, \quad y = \int_0^t h dt \]
4. Relaxation method is a numerical method that is used to solve the laplace getting the solution of an equation that involves the laplace. For example:
   \[ \nabla^2 \phi = F \]

**SELF TEST 2.**

1. Find the \( \mu \) and \( \nu \) at the point I, J, using the linear relationship between horizontal wind and divergence, vorticity deformation type I and II. The \( \mu \) and \( \nu \) at the grid
point (I-1, J), (I, J - 1), (I+1, J), (I, J+1) are (-12,4), (-6,3), (8,-2), (4,5), Δx = Δy = 200 km.

2. Starting from the equation of geostrophic wind in x, y, p, t coordinate, derive the equation of geostrophic vorticity and write in finite difference form.

3. Show that the stream function of wind is \( \Phi / f \)

4. The velocity potential at the point (I+1,J), (I-1,J), (I,J+1), (I,J-1), (I,J) is 4 x 10^5 m^2 s\(^{-1}\), 2.5 x 10^5 m^2 s\(^{-1}\), 6.0 x 10^5 m^2 s\(^{-1}\), 8.4 x 10^5 m^2, 10.6 x 10^5 m^2 s\(^{-1}\). Find the associated divergent wind and divergent at (I,J) if Δx = Δy = 400 km.

5. The stream function at the points (I,J), (I+1,J), (I,J+1), (I-1,J), (I,J-1) are 4.5 x 10^5 m^2 s\(^{-1}\), 8.2 x 10^5 m^2 s\(^{-1}\), 12.8 x 10^5 m^2 s\(^{-1}\), 16.5 x 10^5 m^2 s\(^{-1}\), 15.6 x 10^5 m^2 s\(^{-1}\). Compute the rotational wind speed and direction at (I, J) and the vorticity if Δx = Δy = 400 km.

6. Derive the equation of the stream function and trajectory for the wind field given by

\[ u = U \]

\[ V = V \sin k(x - ct) \]

where \( U \) and \( V \) are constants.

Find the relationship between the amplitude of the streamline and trajectory.

Find the value of \( C \) if

\[ A_s = A_T \]

\[ A_s = 2A_T \]

\[ 2A_s = A_T \]

Where \( A_s \) and \( A_T \) are the amplitude of the streamline and trajectory respectively.
Reference

Prandtl L 1952: Essentials of Fluid Dynamics. Blackie and Son limited
Hare F.K. 1961: The Restless Atmosphere. Hutchinson and Co. LTD
LECTURE 3

DIVERGENCE AND VORTICITY EQUATIONS

3.0 Introduction to Lecture 3

In lecture two, we showed that divergent and vorticity are invariant under rotation of coordinates. For this reason, in some models, the equation of horizontal, the u and v are replaced by the divergent and vorticity. In this lecture we shall derive the divergent and vorticity equation and also see the form they take when approximation is done.

3.1 Objective of Lecture 3

At the end of this lecture, you should be able to

1. Derive the divergence equation starting from the horizontal equation of motion in \(x, y, t\) coordinates.

2. Derive the divergence equation starting from the horizontal equation of motion is \(x, y, p, t\) coordinate.

3. Using use analysis, reduce the divergent equation to the approximation form.

4. State the application of the approximate divergence equation.

5. Rewrite the approximation divergence equation in terms of stream function.

6. Derive the vorticity equation starting from the horizontal equation of motion in \(x, y, z, t\) coordinates

7. Derive the vorticity equation starting from the horizontal equation of motion in \(x, y, p, t\) coordinates.

8. Derive the approximate vorticity equation and state its application.

9. Show the relationship between the rate of change of area and divergence.

3.2 Divergence Equation is \(x, y, z, t\) coordinate

The vector form of the horizontal equation of motion in \(x, y, z, t\) coordinate is
\[
\frac{d\vec{V}}{dt} = -\alpha \nabla P - \hat{f}kX\vec{V} + \vec{F}
\]  
\…………………….. (3.1)

In the component form

\[
\frac{du}{dt} = -\alpha \frac{\partial P}{\partial x} + fv + F_x
\]  
\…………………….. (3.2)
\[
\frac{dv}{dt} = -\alpha \frac{\partial P}{\partial y} + fu + F_y
\]  
\…………………….. (3.3)

Expanding the left hand side of equation 3.2 and 3.3. we obtain

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\alpha \frac{\partial P}{\partial x} + fv + F_x
\]  
\…………………….. (3.4)
\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\alpha \frac{\partial P}{\partial y} - fu + F_y
\]  
\…………………….. (3.5)

Differentiating equation (3.4) w.r.t. \(x\) we obtain

\[
\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial u}{\partial x} \left( \frac{\partial u}{\partial x} \right)^2 + u \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + v \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} \right) + w \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial x} \right) + \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial P}{\partial x}
\]  
\[= -\frac{\partial \alpha}{\partial x} \frac{\partial P}{\partial x} - \frac{\partial \alpha}{\partial x} \frac{\partial P}{\partial x^2} + v \frac{\partial f}{\partial x} + f \frac{\partial v}{\partial x} + \frac{\partial F_x}{\partial x}
\]  
\…………………….. (3.6)

Differentiating equation (3.5) w.r.t. \(y\) we obtain

\[
\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial y} \right) + \frac{\partial v}{\partial y} \left( \frac{\partial v}{\partial y} \right)^2 + u \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right) + v \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial y} \right) + \frac{\partial w}{\partial y} \frac{\partial v}{\partial y} + \frac{\partial P}{\partial y}
\]  
\[= -\frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial y} - \frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial y^2} - u \frac{\partial f}{\partial y} - f \frac{\partial u}{\partial y} + \frac{\partial F_y}{\partial y}
\]  
\…………………….. (3.7)

Adding equation (3.6) and (3.7) we obtain

\[
\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + u \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] + v \frac{\partial}{\partial y} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] + w \frac{\partial}{\partial z} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right]
\]  
\[+ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial v}{\partial y}
\]  
\[= -\frac{\partial \alpha}{\partial x} \frac{\partial P}{\partial x} - \frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial y} - \alpha \left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) + f \left( \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right) + \frac{\partial f}{\partial x} - u \frac{\partial f}{\partial y} + \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}
\]  
\…………………….. (3.8)
notice that

\[
\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 - 2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} + 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = D^2 + 2 J (v, u)
\]

where \( J(v, u) = \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \frac{\partial u}{\partial x} \)

\( J (v, u) \) is called the Jacobian of \( v \) and \( u \)

\[
J(v, u) = \begin{vmatrix}
\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \\
\frac{\partial v}{\partial y} & \frac{\partial u}{\partial y}
\end{vmatrix}
\]

\[
\frac{\partial w}{\partial x} \frac{\partial u}{\partial w} + \frac{\partial w}{\partial y} \frac{\partial v}{\partial w} = \nabla_w \frac{\partial \tilde{v}}{\partial z}
\]

\[
\frac{\partial \alpha}{\partial x} \frac{\partial P}{\partial x} + \frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial y} = \nabla \alpha \cdot \nabla P
\]

\[
f \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = f \zeta
\]

Rewriting equation 3.8 we obtain

\[
\frac{dD}{dt} + D^2 + 2J(v, u) + \nabla_w \frac{\partial \tilde{V}}{\partial z} = -\nabla \alpha \cdot \nabla P - \alpha \nabla^2 P + f \zeta + \frac{\partial f}{\partial x} - u \frac{\partial f}{\partial y} + \nabla \tilde{F}
\]

\[
\text{Term (a) is the rate of change of divergence with time}
\]

\[
\text{Term (b) is the divergent squared}
\]

\[
\text{Term (c) is the jacobian of (v, u)}
\]

\[
\text{Term (d) is the horizontal gradient of vertical motion doted by the vertical shear of wind}
\]

\[
\text{Term (e) is the horizontal gradient of density doted with horizontal gradient of pressure}
\]

\[
\text{Term (f) is the laplacian of pressure}
\]
Term (g) is vorticity time coriolis parameter  
Term (h) and (i) represent the gradient of coriolis parameter, term (i) is zero as the coriolis does not vary along x  
Term (j) is the div of friction  
Using scale analysis we obtain

\[
\frac{dD}{dt} \sim 10^{-12}
\]

\[
D^2 \sim 10^{-12}
\]

\[
2f(v,u) \sim 10^{-10}
\]

\[
\nabla_w \frac{\partial \tilde{v}}{\partial z} \sim 10^{-11}
\]

\[
\nabla \alpha \nabla P \sim 10^{-12}
\]

\[
-\alpha \nabla P \sim 10^{-9}
\]

\[
f\zeta \sim 10^{-9}
\]

\[
u \frac{\partial f}{\partial y} \sim 10^{-9}
\]

We neglect friction and note that \( \frac{\partial f}{\partial x} = 0 \). Retaining the terms with order of magnitude \( 10^{-10} \) or larger we obtain

\[
2J(v,u) = -\alpha \nabla^2 P + f\zeta - u \frac{\partial f}{\partial y}
\]

\[
(3.10)
\]

Equation 3.10 is a diagnostic equation it can be rewritten as

\[
\alpha \nabla^2 P - f\zeta + u\beta + 2J(v,u) = 0
\]

\[
(3.11)
\]

This equation gives us a balance between the wind field and pressure. It is used in the static initialization to adjust either the pressure field or the wind field.

We introduce the stress function

\[
\nabla^2 \psi = \zeta
\]
and if we assume that wind field does not have divergence so that

\[ u = -\frac{\partial \psi}{\partial y} \]

and \[ v = \frac{\partial \psi}{\partial x} \]

Then

\[ u \beta = -\frac{\partial \psi}{\partial y} \frac{\partial f}{\partial y} \]

\[ 2J(v, u) = 2J \left( \frac{\partial \psi}{\partial x}, -\frac{\partial \psi}{\partial y} \right) \]

\[ \alpha \nabla^2 P - f \nabla^2 \psi = \frac{\partial \psi}{\partial y} \frac{\partial f}{\partial y} + 2J \left( \frac{\partial \psi}{\partial x}, -\frac{\partial \psi}{\partial y} \right) = 0 \]

\[ \mbox{.................. (3.12)} \]

Equation 3.12 can be used to compute the pressure field, once the stream function has been obtained.

### 3.3 Divergence Equation in x, y, p, t coordinate

Starting from the equation of horizontal motion

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial P} = -\frac{\partial \phi}{\partial x} + F_x \]

\[ \mbox{................. (3.13)} \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial P} = -\frac{\partial \phi}{\partial y} - F_y \]

\[ \mbox{................. (3.14)} \]

where \[ \omega = \frac{dP}{dt} \]

differentiating equation (3.13) w.r.t. x and equation 3.14 w.r.t y and adding we obtain and following the same steps as in section 3.1 we obtain

\[ \frac{dD}{dt} + D^2 + 2J(v, u) + \nabla \omega \cdot \frac{\partial \omega}{\partial P} = -\nabla^2 \phi + f \zeta + \nu \frac{\partial f}{\partial x} - u \frac{\partial f}{\partial y} + \nabla \vec{F} \]

\[ \mbox{........... (3.15)} \]

Considering the terms with the highest order of magnitude we obtain
\[ \nabla^2 \phi - \zeta f + u\beta + 2J(v, u) = 0 \]

(3.16)

Again equation 3.16 is a diagnostic equation.

If the jacobian term is dropped we obtain

\[ \nabla^2 \phi - \zeta f + u\beta = 0 \]

............... (3.17)

We recall that

\[ \nabla^2 \psi = \zeta \]

\[ \nabla^2 \phi - f \nabla^2 \psi - \beta \frac{\partial \psi}{\partial y} = 0 \]

............... (3.20)

Using equation (3.20) we can obtain the geopotential from the stream function.

**Activity 3.1**

1. Starting from equations 3.13 and 3.14 show all the steps that will enable you to arrive at equation 3.15.

---

**3.4 The Vorticity Equation in x, y, z, t Coordinates**

Starting from the horizontal equation

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\alpha \frac{\partial P}{\partial x} + fu + F_x
\]

3.4

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\alpha \frac{\partial P}{\partial y} - fu + F_y
\]

3.5

We differentiate equation 3.4 w.r.t. y and equation 3.5 w.r.t. x we obtain

\[
\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial y} \right) + \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + u \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} + v \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) + \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} + w \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial y} \right)
\]
\[
\begin{align*}
\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) & + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \left( \frac{\partial u}{\partial y} \right) + v \left( \frac{\partial v}{\partial x} \right) + w \left( \frac{\partial v}{\partial z} \right) + \frac{\partial w}{\partial x} v
\end{align*}
\]
\[= - \frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial y} - \alpha \frac{\partial^2 P}{\partial y \partial x} - u \frac{\partial f}{\partial x} v + \frac{\partial F_y}{\partial x} + \frac{\partial F_z}{\partial x} \]
\[= - \frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial y} - \alpha \frac{\partial^2 P}{\partial y \partial x} - \frac{\partial f}{\partial x} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \]
\[\text{Subtracting 3.21 from 3.22 we obtain}
\]
\[\frac{\partial}{\partial t} \left[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \left( \frac{\partial u}{\partial y} \right) + v \left( \frac{\partial v}{\partial x} \right) + w \left( \frac{\partial v}{\partial z} \right) + \frac{\partial w}{\partial x} v
\]
\[= - \frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial y} - \alpha \frac{\partial^2 P}{\partial y \partial x} - \frac{\partial f}{\partial x} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \]
\[= - \frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial y} - \alpha \frac{\partial^2 P}{\partial y \partial x} - u \frac{\partial f}{\partial x} \frac{\partial v}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \]
\[\text{Note the following}
\]
\[\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]
\[\mathbf{k} \left( \nabla \times \frac{\partial v}{\partial z} \right) = \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial x} \]
\[\mathbf{k} \nabla \alpha \times \nabla P = \frac{\partial \alpha}{\partial x} \frac{\partial P}{\partial y} + \frac{\partial \alpha}{\partial y} \frac{\partial P}{\partial x} \]
\[\mathbf{k} \nabla \times \mathbf{F} = \frac{\partial F_y}{\partial x} - \frac{\partial F_z}{\partial y} \]
\[\frac{df}{dt} = - \frac{\partial f}{\partial t} - u \frac{\partial f}{\partial x} - v \frac{\partial f}{\partial y} - w \frac{\partial f}{\partial z} \]
\[-\frac{df}{dt} = -\nu \frac{\partial f}{\partial y}\] ……………………… (3.29)

Since \(\frac{\partial f}{\partial t} = 0\), \(\frac{\partial f}{\partial x} = 0\) and \(\frac{\partial f}{\partial z} = 0\)

Rewriting 3.23 using 3.24 to 3.29 we obtain

\[\frac{d}{dt} \zeta + \zeta D + \hat{k} \nabla w \times \frac{\partial \hat{V}}{\partial z} = -k \nabla \alpha \times \nabla P - fD - \frac{df}{dt} + k \nabla \times \vec{F} \] ………… (3.30)

(a) \hspace{1cm} (b) \hspace{1cm} (c) \hspace{1cm} (d) \hspace{1cm} (e) \hspace{1cm} (f) \hspace{1cm} (g)

term (a) is the rate of change of relative vorticity
term (b) is the product of vorticity and divergence
term (c) is the twisty term, which is derived from the cross product of horizontal gradient of vertical velocity and vertical wind shear.
term (d) is the solenoidal term
term (e) is the product of coriolis and divergence
term (f) is the rate of change of coriolis term with
term (g) is the frictional term.

Equation (3.30) can be reorganized to take the form

\[\frac{d\eta}{dt} + \eta D + \hat{k} \nabla w \times \frac{\partial \hat{V}}{\partial z} + \hat{k} \nabla \alpha \times \nabla P - \hat{k} \nabla \times \vec{F} = 0 \] ………………… (3.31)

where \(\eta = \zeta + f\)

The last three terms are of lower magnitude compared to the first two therefore they can be dropped yielding

\[\frac{d\eta}{dt} + \eta D = 0 \] ………………… (3.32)

Equation (3.32) is the approximate vorticity equation and gives the relationship between the rate of change of absolute vorticity. It is used to determine the contribution of jet-stream to the divergence.

We can also rewrite as

\[\frac{1}{\eta} \frac{d\eta}{dt} = -D \] ………… (3.34a)
or \[ \frac{d}{dt}(ln \eta) = D \] \hspace{1cm} (3.34b)

For mid latitude it can further be shown that

\[ \frac{d\eta}{dt} + fD = 0 \] \hspace{1cm} (3.35)

Moreover the first term of equation 3.35 is still larger than the second which reduces the equation to

\[ \frac{d\eta}{dt} = 0 \] \hspace{1cm} (3.36)

or

\[ \frac{\partial \eta}{\partial t} + \mathbf{u} \cdot \nabla \eta + v \frac{\partial \eta}{\partial y} + w \frac{\partial \eta}{\partial z} = 0 \] \hspace{1cm} (3.37)

and the last term on the left of 3.37 is small so that

\[ \frac{\partial \eta}{\partial t} + \mathbf{V}_u \cdot \nabla \eta = 0 \] \hspace{1cm} (3.38)

In the atmosphere the motion is largely zonal so that \( v \frac{\partial \eta}{\partial y} \) can also be dropped and we have

\[ \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} = 0 \] \hspace{1cm} (3.39)

but \[ \frac{\partial f}{\partial t} = 0 \] and \[ \frac{\partial f}{\partial x} = 0 \] so that 3.39 reduces to

\[ \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} = 0 \] \hspace{1cm} (3.40)

Equation 3.40 is the simplified vorticity equation. It forms a simple model which is used to study some features in numerical weather prediction.
3.5 The Vorticity Equation in x, y, p, t Coordinates

Starting from the horizontal equation of motion in x, y, p, t coordinate

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial P} = -\frac{\partial \phi}{\partial x} + f v + F_x \quad \text{................. (3.13)}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial P} = -\frac{\partial \phi}{\partial y} - f u + F_y \quad \text{................. (3.14)}
\]

Differentiating equation 3.13 w.r.t. y and equation 3.14 w.r.t. x and following similar steps as section 3.3, we obtain

\[
\frac{d\eta}{dt} + \eta D + \hat{k} \cdot \nabla \omega \frac{\partial \tilde{v}}{\partial P} - \hat{k} \cdot \nabla F = 0 \quad \text{............... (3.41)}
\]

**Activity 3.3**

1. Starting from equations 3.13 and 3.14, follow all steps as done in section to obtain equations 3.41

2. Compare equation 3.36 and 3.31, why do you think the solenoid term is missing in equation 3.41.

Equation 3.41 had an advantage to equation 3.40 in that the density does not have to be considered.

As noted earlier, when terms of lower magnitude are dropped from equation 3.40 we obtain

\[
\frac{d\eta}{dt} + \eta D = 0
\]

Or

\[
\frac{d\eta}{dt} = -\eta D \quad \text{3.42}
\]
Later on we will show that

\[ D = -\frac{\partial \omega}{\partial \rho} \]  \hspace{1cm} (3.43)

So that equation 3.37 becomes

\[ \frac{d \eta}{dt} = \eta \frac{\partial \omega}{\partial \rho} \]

in mid-latitude \( \zeta << f \) so that

\[ \frac{d \eta}{dt} = f \frac{\partial \omega}{\partial \rho} \] \hspace{1cm} (3.44)

Equation 3.39 is used in simple prediction models.

### 3.5 Relationship Between Divergence and the Rate of Change of Area

Let the area element to \( \delta A \) be

\[ \delta A = \delta x \delta y \] \hspace{1cm} (3.45)

taking log

\[ \ln \delta A = \ln \delta x + \ln \delta y \] \hspace{1cm} (3.46)

Differentiating w.r.t. time

\[ \frac{1}{\delta A} \frac{d}{dt} \delta A = \frac{1}{\delta x} \frac{d}{dt} (\delta x) + \frac{1}{\delta y} \frac{d}{dt} (\delta y) \] \hspace{1cm} (3.47)

\[ \frac{d}{\delta x} \frac{d}{dt} (\delta x) = \frac{1}{\delta x} \frac{d}{dt} (\delta x) = \frac{\dot{u}}{\dot{x}} \] \hspace{1cm} (3.48)

\[ \frac{d}{\delta y} \frac{d}{dt} (\delta y) = \frac{1}{\delta y} \frac{d}{dt} (\delta y) = \frac{\dot{v}}{\dot{y}} \] \hspace{1cm} (3.49)
\[
\frac{1}{\delta A} \frac{d(\delta A)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
\]

Thus, divergence in equal to the change in area of the fluid element per unit time and per unit area.

Alternative method of deriving equation 3.45 is to consider a case of a flow that is purely divergence and emanating from or flowing towards the centre of a circle of radius \( r \).

From lecture 2 we showed that for a purely divergent motion

\[ u = \frac{1}{2} D x \quad \text{..........} \quad 3.51 \]

and

\[ v = \frac{1}{2} D y \quad \text{..........} \quad 3.52 \]

If we combine 3.527 and 3.54 we obtain

\[
v = \frac{1}{2} D r \sin \theta \quad \text{..........} \quad 3.55
\]

differentiating 3.54 w.r.t time and since the air parcel moves in a straight line the direction does not change with time hence \( \theta \) is constant.

We obtain
\[
v = \frac{dy}{dt} = \frac{dr}{dt} \sin \theta \quad \text{.......................... 3.56}
\]

Substituting 3.56 into 3.55

We obtain

\[
\frac{dr}{dt} \sin \theta = \frac{1}{2} D r \sin \theta \quad \text{.......................... 3.57}
\]

which gives

\[
\frac{dr}{dt} = \frac{1}{2} D r \quad \text{.......................... 3.58}
\]

The area of a circle of radius \( r \) is

\[
A = \pi r^2 \quad \text{.......................... 3.59}
\]

Differentiating w.r.t. time

\[
\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \quad \text{.......................... 3.60}
\]

or

\[
\frac{dr}{dt} = \frac{1}{2\pi r} \frac{dA}{dt} \quad \text{.......................... 3.61}
\]

Substituting 3.61 into 3.58 we obtain

\[
\frac{1}{2\pi r} \frac{dA}{dt} = \frac{1}{2} D r
\]

or

\[
\frac{1}{\pi r^2} \frac{dA}{dt} = D
\]

which is clearly

\[
\frac{1}{A} \frac{dA}{dt} = D
\]

Prandtl L 1952: Essentials of Fluid Dynamics. Blackie and Son limited
Hare F.K. 1961: The Restless Atmosphere. Hutchinson and Co. LTD
LECTURE FOUR
CIRCULATION AND OUTFLOW

4.0 Introduction

In lecture two we considered the kinematics of the atmospheric flow. We saw how the motion is related to the four derived parameters, namely divergence, vorticity, deformation type I and deformation type II. In lecture three we studied the various forces that lead to the rate of change vorticity and divergence. In this lecture we will give further interpretation to vorticity by seeing how it is related to circulation and interpret divergence in terms of outflow.

4.1 Objectives of the Lecture

At the end of this lecture, you should be able to
1. State and prove Kelvins theorem of circulation.
2. Show the relationship between Kelvins theorem of circulation and Bjerkrest theorem.
3. Show the relationship between circulation and vorticity.
4. State the forces that contribute to the rate of change of circulation.
5. Differentiate between direct and indirect circulation.
6. Demonstrate the application of circulation to land and sea breeze, mountain and valley wind, monsoons.
7. Show the relationship between outflow and divergent.
8. Use the relationship between circulation and vorticity to derive the vorticity in spherical coordinate.
9. Use the relationship between outflow and divergence to derive the divergence in spherical coordinate.

4.2 Circulation

If $\vec{V}$ is an arbitrary velocity field, and if we let $Q$ and $R$ be two arbitrary points in three dimensional space connected by an arbitrary curve.

The integral (along the curve)

$$F = \int_{Q}^{R} \vec{V}.d\vec{r}$$

(4.1)
where $d\vec{r}$ is an infinitesimal vector along the curve, is called the flow from $Q$ to $R$.

We note that

$$\vec{V}.d\vec{r} = V_r dr$$  \hspace{1cm} (4.2)$$

Where $V_r$ is the component of $\vec{V}$ along this direction given by $d\vec{r}$.

Note that the integral $F$ defined in equation 4.1 will depend in general upon the curve from $Q$ to $R$ which is considered in the evaluation.

The definition of the integral given in equation 4.1 may be general for any vector, rather than the flow. We may consider the curve integral of any vector $\vec{\rho}$.

$$G = \int_Q^R \vec{\phi}.d\vec{r}$$  \hspace{1cm} (4.3)$$

If we consider a case where the vector $\vec{\phi}$ is the ascendant of some field $P$ thus

$$\vec{\phi} = \nabla P$$  \hspace{1cm} (4.4)$$

$$\vec{\rho} = \frac{\partial P}{\partial x} \hat{i} + \frac{\partial P}{\partial y} \hat{j} + \frac{\partial P}{\partial z} \hat{k}$$  \hspace{1cm} (4.5)$$

and

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$  \hspace{1cm} (4.6)$$

Then

$$\vec{\rho}.d\vec{r} = \left( \frac{\partial P}{\partial x} \hat{i} + \frac{\partial P}{\partial y} \hat{j} + \frac{\partial P}{\partial z} \hat{k} \right) \left( dx \hat{i} + dy \hat{j} + dz \hat{k} \right)$$

$$\vec{\rho}.dr = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz$$  \hspace{1cm} (4.7)$$

$$\vec{\rho}.dr = dP$$  \hspace{1cm} (4.8)$$

We therefore have

$$\int_Q^R \vec{\rho}.d\vec{r} = \int_Q^R \nabla P.d\vec{r} = \int_Q^R dp = P_R - P_Q$$  \hspace{1cm} (4.9)$$
Now consider the special case where the curve is a closed curve.

The integral in 4.1 is called the circulation, we may therefore define circulation as the flow along a closed curve.

\[
C = \oint_Q \vec{V} \cdot d\vec{r} = \oint C \vec{V} \cdot d\vec{r} = \oint \vec{V} \cdot d\vec{r}
\] (4.10)

This concept may naturally be generalized to apply to any vector field \( \vec{\rho} \), and we normally talk about the circulation of \( \vec{\rho} \) as the integral.

\[
C = \oint \vec{\rho} \cdot d\vec{r} = \oint \rho \cdot dr
\] (4.11)

It can be seen that the circulation of an ascendant is zero along any curve.

\[
\oint \nabla P \cdot d\vec{r} = \oint dP = \int^0_P dP = P_f - P_i = 0
\] (4.12)

### 4.3 Kelvin’s Circulation Theorem

We define the acceleration of the circulation as

\[
a_c = \frac{dc}{dt}
\] (4.10)

and the circulation of the acceleration as

\[
C_{acc} = \oint \vec{\dot{r}} \cdot d\vec{r}
\] (4.11)

Kelvin’s circulation theorem states that the acceleration of the circulation is equal to the circulation of the acceleration thus

\[
\frac{dc}{dt} = \int \frac{d\vec{r}}{dt} \cdot d\vec{r}
\] (4.12)

Proof:
\[ C = \oint \vec{V} \cdot d\vec{r} \]

\[ \frac{dc}{dt} = \frac{d}{dt} \left[ \oint \vec{V} \cdot d\vec{r} \right] \]

\[ = \oint \frac{d\vec{V}}{dt} \cdot d\vec{r} + \oint \vec{V} \cdot \frac{d(d\vec{r})}{dt} \]  

(4.13)

Consider the second term on the right hand side of equation 4.13

\[ \oint \vec{V} \cdot \frac{d(d\vec{r})}{dt} = \oint \vec{V} \cdot \left( \frac{d\vec{r}}{dt} \right) \]

\[ = \oint \vec{V} \cdot d\vec{V} \]  

(4.14)

\[ = \oint \frac{1}{2} d(\vec{V} \cdot \vec{V}) \]

\[ = 0 \]

Hence

\[ \frac{dc}{dt} = \oint \frac{d\vec{V}}{dt} \cdot d\vec{r} \]  

(4.15)

which completes the proof of Kelvin’s circulation theorem.

4.4 Bjerknes’ Circulation Theorem

Consider the equation of motion in the free atmosphere, where frictional force may be neglected.

then

\[ \frac{d\vec{V}}{dt} = -\alpha \nabla p - 2\Omega \times \vec{V} + \vec{g} \]  

(4.16)

If we integrate equation 4.16 along a closed curve we obtain the following.

\[ \oint \frac{d\vec{V}}{dt} \cdot d\vec{r} = \oint -\alpha \nabla p \cdot d\vec{r} - \oint \left( 2\Omega \times \vec{V} \right) \cdot d\vec{r} + \oint \vec{g} \cdot d\vec{r} \]  

(4.17)

From Kelvin’s theorem, the left hand side is the acceleration of the circulation.
\[ \oint \frac{d\mathbf{V}}{dt} \cdot d\mathbf{r} = \frac{dc}{dt} \]

The first term in the right of equation (4.17) is

\[ \oint -\alpha \nabla p \cdot d\mathbf{r} = \oint -\alpha (\nabla p \cdot d\mathbf{r}) \]

\[ = \oint -\alpha dp \]

We define geopotential \[ \phi = g z \] (4.18)

\[ \nabla \phi = \frac{\partial (g z)}{\partial x} \hat{i} + \frac{\partial (g z)}{\partial y} \hat{j} + \frac{\partial (g z)}{\partial z} \hat{k} \] (4.19)

In the x,y,z coordinates z is independent of x and y hence

\[ \nabla \phi = g \hat{k} \]

\[ \nabla \phi = \bar{g} \] (4.20)

The third term on the right hand side of equation 4.17 becomes

\[ \oint \bar{g} \cdot d\mathbf{r} = \oint \nabla \phi \cdot d\mathbf{r} \]

\[ = \oint d\phi \]

\[ = 0 \] (4.21)

We have therefore

\[ \frac{dc}{dt} = -\oint \alpha dp - \oint 2(\bar{\Omega} \times \mathbf{V}) \cdot d\mathbf{r} \] (4.22)

Equation 4.22 is the circulation theorem of Bjerknes.

### 4.5 Relationship Between Vorticity and Circulation

Consider a horizontal flow with velocity \( \mathbf{V} \)

Where

\[ \mathbf{V} = u \hat{i} + v \hat{j} \] (4.23)

If the flow is over a horizontal surface, the position vector \( \mathbf{r} \) is defined by
\[
\vec{r} = x \hat{i} + y \hat{j}
\]  
(4.24)

or
\[
d\vec{r} = dx \hat{i} + dy \hat{j}
\]  
(4.25)

The circulation over the horizontal is given by
\[
C = \oint \vec{V} \cdot d\vec{r}
\]  
(4.26)

Stokes theorem states that the line integral over a closed curve of a vector \( \vec{F} \) is equal to the area integral of the curve of that vector.

Thus
\[
\oint \vec{F} \cdot d\vec{r} = \int_A (\nabla \times \vec{F}) \cdot d\vec{A}
\]  
(4.27)

In equation 4.27 we
Let \( \vec{F} = \vec{V} \)

And \( d\vec{A} = dx\,dy \, \hat{k} \) the area vector.

Then
\[
C = \oint \vec{V} \cdot d\vec{r} = \int_A (\nabla \times \vec{V}) \, dx\,dy \, \hat{k}
\]

\[
= \int_A \hat{k} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \, dx\,dy \, \hat{k}
\]

\[
C = \int_A \zeta \, dA
\]  
(4.28)

Thus a change in \( C \) is given by
\[
dC = \zeta \, dA
\]  
(4.29)

which can be rewritten as
\[
\zeta = \frac{dC}{dA}
\]  
(4.30)

Thus vorticity in circulation around a unit area. Or circulation is the area integral of vorticity.
Alternative approach:

\[
\delta C = \Sigma V_i \Delta r_i \quad (4.31)
\]

where \( V_i \) is the parallel wind component to \( \Delta r_i \)

\[
\delta C = \frac{1}{2} \left[ u + u + \frac{\partial u}{\partial x} \delta x \right] \delta x
\]

\[
+ \frac{1}{2} \left[ v + \frac{\partial v}{\partial x} \delta x + v + \frac{\partial v}{\partial y} \delta y \right] \delta y
\]

\[
- \frac{1}{2} \left[ u + \frac{\partial u}{\partial y} \delta y + u + \frac{\partial u}{\partial x} \delta x \right] \delta x
\]

\[
- \frac{1}{2} \left[ v + \frac{\partial v}{\partial y} \delta y + v \right] \delta y \quad (4.32)
\]

Evaluating equation (4.32) we obtain

\[
\delta C = \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta x \delta y \quad (4.33)
\]
or \[ dC = \zeta \, dA \]

\[ C = \int_A \zeta \, dA \]

### 4.6 Interpretation of the terms in Bjerkness Theory of Circulation

Recall equation 4.22

\[
\frac{dc}{dt} = -\oint_A \alpha \, dp - \oint_A 2(\mathbf{\Omega} \times \mathbf{v}) \cdot d\mathbf{r} \tag{4.22}
\]

Consider term I, rewriting it in its original form.

\[-\oint_A \alpha \, dp = -\oint_A \alpha \, \nabla p \cdot d\mathbf{r} \tag{4.33}\]

Applying stokes theorem to the right hand side of equation 4.33, we have

\[-\oint_A \alpha \, \nabla p \cdot d\mathbf{r} = -\int_A (\nabla \times \alpha \, \nabla p) \cdot d\mathbf{A} \]

\[= -\int_A (\nabla \alpha \times \nabla p) \cdot d\mathbf{A} - \int_A \alpha (\nabla \times \nabla p) \cdot d\mathbf{A} \]

The second term on the right hand side of equation 4.34 is zero since curl of Grad = 0

\[-\oint_A \alpha \, \nabla p \cdot d\mathbf{r} = -\int_A (\nabla \alpha \times \nabla p) \cdot d\mathbf{A} \]

Is called the \( \mathbf{S} = \nabla \alpha \times (-\nabla p) \) solenoid vector. \( \tag{4.35} \)

### 4.7 Application of the Circulation Theorem

Consider equation 4.48. We know that when a parcel of air is at rest, the Coriolis force acting on it is zero. So that when a parcel of a starting from rest, the forces that will contribute to the change in circulation is the solenoid. On the other hand, if we have motion in a field where the isosteric lines are parallel to the isobars, then the contribution
of the solenoid term will be zero and the change in circulation will depend only on the
coriolis effect.

**Activity 4.1**

1. Show that the solenoidal vector \( \vec{S} = -\nabla \alpha \times \nabla \rho \) may be written as
   
   (i) \( -\frac{R}{P} \nabla T \times \nabla P \)
   
   (ii) \( \frac{C}{\theta} \nabla T \times \nabla \theta \)

2. Why is it often desirable to use temperature and potential temperature in the
   vertical cross-sections.

---

**4.8 Application of the Solenoid Vector to Land and See Breeze**

As an application where we may neglect the coriolis effect initially we consider the
phenomenon of the sea breeze. Let us assume that the air at a certain time is in
equilibrium over an underlying surface consisting partly of ocean and partly of land (see
figure). In the equilibrium state the isobaric and isostatic surface are both horizontal.
During the day we have a stronger heating of the land than of the ocean. The lower layer
of the atmosphere over land will be heated and the specific volume will be larger. The
isostatic surfaces will tilt as indicated in Figure 4.9 Along the physical curve (dashed
line in Figure 4.9) we will have a circulation in the direction from \( \nabla \alpha \) to \(-\nabla \rho \), i.e. from
ocean to land in the lower layer and from land to ocean (return current) in the upper layer.

The flow is in the same direction as the solenoid vector \( \vec{S} \). Such a flow is said to be a
direct flow.

**Activity 4.2**

1. Using the similar procedure as demonstrated above (section 4.b) show that
   this land breeze is a direct flow.

2. Apply a similar principle as for the case of sea and land breeze to the
   anabatic and katabatic flow.
As the circulation increases in intensity we will get a mass transport of cold air below it and warm air above. The isobaric surfaces will start to tilt so that they are no longer horizontal.

You recall from what you learnt in synoptic meteorology that the geopotential thickness is proportion to the near temperature of a given layer. Once the speed increases we cannot ignore the coriolis effect. If the coast is oriented from south to north, we will initially have a see breeze from the east (from the above figure) in the lower layers. However as the day progresses, we can expect a change to a south-easterly direction. If the flow is in the northern hemisphere to a orth-easterly if the flow is in the southern hemisphere. (Recall that the coriolis force deflects the particle to the right of motion in the northern hemisphere and to the left of motion in the southern hemisphere).

During the night we have reversed relation, the lower layers of the atmosphere over land will be cooled more rapidly than over the ocean. The tilt of the isosteric surfaces relative to the isobaric surfaces will be opposite of the tilt indicated in Figure 4.9b. and we will get a land breeze.

4.9 Illustration of the Effect of Coriolis Effect on the Circulation

(a) Circulation around a physical curve coinciding in a latitude circle.

If the curve is displaced toward the equator, we will get an increase in $A_E$ and therefore a decrease in the circulation, or in other words, a tendency for the creation of an anticyclonic circulation.

Hence

$$-2\Omega \frac{dA_E}{dt} < 0$$

(4.49)

If the displacement were towards the pole, the area would decrease, and we would get an increase in cyclonic circulation.

$$\frac{dA_E}{dt} < 0 \text{ hence } -2\Omega \frac{dA_E}{dt} > 0$$

(4.50)

(b) Horizontal physical curve of constant area
\[ A_{\varphi_1} = A_{\varphi_2} \]
\[ A_{\varphi_{1E}} = A_{\varphi_1} \sin \varphi_1 \]
\[ A_{\varphi_{2E}} = A_{\varphi_2} \sin \varphi_2 \]

since \( \varphi_2 > \varphi_1 \)
\[ A_{\varphi_2} > A_{\varphi_1} \]

If we consider a horizontal physical curve of constant area. If such a curve moves towards the pole we get an increase i.e. \( A_E \) and therefore a creation of anticyclonic circulation, while the opposite effect (increase in cyclonic circulation) is observed when the curve moves towards the equator.

(c) Horizontal curve at fixed latitude

We note finally that if we have horizontal curve which remains at a given latitude, contraction will create positive (cyclonic) circulation, while expansion leads to creation of negative (anticyclonic) circulation.

\[ \frac{dA}{dt} > 0 \quad \Rightarrow \quad \text{decrease of cyclonic circulation or increase of anticyclonic circulation} \]

\[ \frac{dA}{dt} < 0 \quad \Rightarrow \quad \text{increase of cyclonic circulation or decrease of anticyclonic circulation} \]

4.10 Vorticity of the motion of solid rotation

Consider the circulation around a circle of radius \( r \) in a velocity field corresponding to a rotation with a constant velocity \( \omega \).

\[ d\vec{r} \approx r \, d\theta \, \hat{T} \quad \text{where} \ \hat{T} \ \text{is the tangent to the circle.} \]
\[
\mathbf{V} = \frac{d\mathbf{r}}{dt}
\]
\[
= r \frac{d\theta}{dt} \mathbf{\hat{T}}
\]
\[
\mathbf{V} = r \omega \mathbf{\hat{T}} \quad (4.51)
\]

From the definition of the circulation
\[
C = \oint \mathbf{V} \cdot d\mathbf{r} \quad (4.52)
\]

We substitute for \( \mathbf{V} \) and \( d\mathbf{r} \) we obtain
\[
C = \oint r \omega \mathbf{\hat{T}} \cdot r d\theta \mathbf{\hat{\theta}}
\]
\[
= \oint r^2 \omega \ d\theta
\]
\[
= \int_0^{2\pi} r^2 \omega \ d\theta \quad (4.53)
\]

on integrating we obtain \( C = 2\pi r^2 \omega \). Now recall that the area of a circle is given by \( \pi r^2 \).
So if we divide both sides of equation 4.53, by the area we obtain

\[
\frac{C}{A} = \frac{2\pi r^2 \omega}{\pi r^2} = \frac{2\pi r^2 \omega}{A}
\]

or

\[
\frac{C}{A} = \frac{2\pi r^2 \omega}{\pi r^2}
\]

\[
\frac{C}{A} = 2\omega \quad (4.53)
\]

The left hand side of equation 4.53 is circulation divided by area, which we showed earlier that it is equal to vorticity.

Thus \( \zeta = \frac{C}{A} \)

Hence
\[ \zeta = 2 \omega \] (4.54)

From equation 4.54 we see that the vorticity of the motion of solid rotation is equal twice the angular velocity.

In lesson one we stated that \( \zeta \) is the vertical vorticity. If we consider the earth as a solid rotation, we can resolve the rotation vector \( \vec{\Omega} \) into three components.

\[ \vec{\Omega} = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k} \]

Consider a point on the earth’s surface at latitude \( \varphi \). \( \vec{\Omega} \) is in the plane of \( \hat{j} \) and \( \hat{k} \) and therefore its component in the \( \hat{i} \) direction is \( \Omega \) is zero.

Therefore

\[ \vec{\Omega} = \Omega \hat{i} + \Omega \cos \varphi \hat{j} + \Omega \sin \varphi \hat{k} \] (5.56)

If we consider the vertical vorticity due to the rotation of the earth, \( \zeta_k \) from equation 4.54, we see that it is \( 2 \Omega \sin \varphi \).

Thus

\[ \zeta_k = 2\Omega \sin \varphi \] (5.57)

OR

\[ \zeta_k = f \]
Thus the coriolis parameter is the vertical vorticity due to the rotation of the earth. The sum of the relative vorticity $\zeta$ and the vorticity due to the rotation of the earth $f$ is referred to as the absolute vorticity $\eta$.

$$\eta = \zeta + f$$  \hspace{1cm} (5.58)

### 4.11 Vorticity in Spherical Coordinates

Since the earth is a sphere, it is necessary to write the equations in spherical coordinate so that they are applicable to the actual situation in the atmosphere. We will use what we have learnt about the relationship between relative vorticity $\zeta$ and circulation $C$ to represent relative vorticity in spherical coordinate.

Recall that

$$\zeta = \frac{dC}{dA}$$

Let us consider the circulation along the curve ABCD.

We consider the displacements in the $i$-direction to be eastward and in the $j$-direction to be positive if it is directed northward.

From the above figure the area is

$$dA = dx dy$$

$$dx = a \cos \phi \, \delta \lambda$$

$$dy = a d\phi$$  \hspace{1cm} hence

$$dA = a^2 \cos \phi \, \delta \lambda \, \delta \phi$$  \hspace{1cm} (4.59)

Let the velocity component at the centre be $u_o, v_o$ and the velocity components at the points A, B, C and D to be $u_A, v_A; \ u_B, v_B; \ u_C, v_C$ and $u_D, v_D$ respectively. We will be interested in the velocity components mid-way, thus at the points $A', B', C'$ and $D'$ which will be taken as the mean along each side.

$$u_{A'} = u\left(\lambda, \phi - \frac{1}{2} \delta \phi\right)$$

$$v_{A'} = v\left(\lambda, \phi - \frac{1}{2} \delta \phi\right)$$
\[ u_{\lambda'} = u \left( \lambda + \frac{1}{2} \delta \lambda \phi \right) \]

\[ v_{\lambda'} = v \left( \lambda + \frac{1}{2} \delta \lambda \phi \right) \]

\[ u_{\varphi'} = u \left( \lambda, \varphi + \frac{1}{2} \delta \varphi \right) \]

\[ v_{\varphi'} = v \left( \lambda, \varphi + \frac{1}{2} \delta \varphi \right) \]

\[ u_{\phi'} = u \left( \lambda - \frac{1}{2} \delta \lambda \phi \right) \]

\[ v_{\phi'} = v \left( \lambda - \frac{1}{2} \delta \lambda \phi \right) \] (5.60)

Can you recall taylors expansion series we use in lecture one. If you have forgotten, go back and review. If we apply taylors expression to the above equations, and retaining only the first two terms we have.

\[ u_{\lambda'} = u_{\theta} - \frac{1}{2} \frac{\partial u}{\partial \phi} \delta \phi \]

\[ v_{\lambda'} = v_{\theta} - \frac{1}{2} \frac{\partial v}{\partial \phi} \delta \phi \]

\[ u_{\varphi'} = u_{\theta} + \frac{1}{2} \frac{\partial u}{\partial \lambda} \delta \lambda \]

\[ v_{\varphi'} = v_{\theta} + \frac{1}{2} \frac{\partial v}{\partial \lambda} \delta \lambda \]

\[ v_{\varphi'} = v_{\theta} + \frac{1}{2} \frac{\partial u}{\partial \varphi} \delta \varphi \]

\[ v_{\varphi'} = v_{\theta} + \frac{1}{2} \frac{\partial v}{\partial \varphi} \delta \varphi \]
$$u_{\mu'} = u_o - \frac{1}{2} \frac{\partial u}{\partial \lambda} \delta \lambda$$

$$v_{\nu'} = v_o - \frac{1}{2} \frac{\partial v}{\partial \lambda} \delta \lambda$$  \hspace{1cm} (5.61)

The circulation along the curve $A B C D$ is given by

$$dc = \oint V_i d\ell$$ \hspace{1cm} (5.62)

Where $V_i$ is the tangential velocity along the curve. Since we have four sides we can turn the integral into summation.

$$dc = \sum_{i=1}^{4} V_{ni} d\ell_i$$ \hspace{1cm} (5.63)

If we start from A, we can name side AB as 1, BC 2 etc.

$$dc = V_{1i} d\ell_1 + V_{2i} d\ell_2 + V_{3i} d\ell_3 + V_{4i} d\ell_4$$ \hspace{1cm} (5.64)

$$V_{1i} = u_{\lambda} = u_o - \frac{1}{2} \frac{\partial u}{\partial \phi} \delta \phi$$ \hspace{1cm} (5.65)

$$d\ell_1 = a \cos \left( \phi - \frac{1}{2} \delta \phi \right) \delta \lambda$$ \hspace{1cm} (5.66)

$$V_{2i} = v_{\nu} = v_o + \frac{1}{2} \frac{\partial V}{\partial \lambda} \delta \lambda$$ \hspace{1cm} (5.67)

$$d\ell_2 = a \delta \phi$$ \hspace{1cm} (5.68)

$$V_{3i} = u_o + \frac{1}{2} \frac{\partial u}{\partial \phi} \delta \phi$$ \hspace{1cm} (5.69)

$$d\ell_3 = -a \cos \left( \phi + \frac{1}{2} \delta \phi \right) \delta \lambda$$ \hspace{1cm} (5.70)

(Negative because it is to the west)

$$V_{4i} = v_o - \frac{1}{2} \frac{\partial v}{\partial \lambda} \delta \lambda$$ \hspace{1cm} (5.71)
\[ d\ell = -a \delta \varphi \]  \hspace{1cm} \text{(5.72)} \\
\text{(Negative because it is to the south)}

Combining we have

\[
dc = \left( u_v - \frac{1}{2} \frac{\partial u}{\partial \varphi} \delta \varphi \right) a \cos(\varphi - \frac{1}{2} \delta \varphi) \partial \lambda + \left( v_v + \frac{1}{2} \frac{\partial v}{\partial \lambda} \delta \lambda \right) a \delta \varphi \\
- \left( u_v + \frac{1}{2} \frac{\partial u}{\partial \varphi} \delta \varphi \right) a \cos(\varphi + \frac{1}{2} \delta \varphi) \partial \lambda - \left( v_v - \frac{1}{2} \frac{\partial v}{\partial \lambda} \delta \lambda \right) a \delta \varphi \hspace{1cm} \text{(4.72)}
\]

Simplifying equation 4.72 and using the trigonometry identity we obtain

\[
dc = \frac{\partial v}{\partial \lambda} a \delta \lambda \delta \varphi - \frac{\partial u}{\partial \varphi} \cos \left( \frac{1}{2} \delta \varphi \right) a \cos \varphi \partial \lambda \partial \varphi + 2u \sin \left( \frac{1}{2} \delta \varphi \right) a \sin \varphi \partial \lambda \\
(4.73)
\]

Activity 4.3

Before you continue simplify equation 4.72 to obtain 4.73

If we divide equation 4.730 by the area we obtain

\[
\zeta = \frac{dc}{dA} = \frac{\partial v}{\partial \lambda} \frac{a \delta \lambda \delta \varphi}{a^2 \cos \varphi \partial \lambda \partial \varphi} - \frac{\partial u}{\partial \varphi} \cos \left( \frac{1}{2} \delta \varphi \right) \frac{a \cos \varphi \partial \lambda \partial \varphi}{\left( a^2 \cos \delta \varphi \partial \varphi \right)} \\
+ \frac{2u \sin \left( \frac{1}{2} \delta \varphi \right) a \sin \varphi \partial \lambda}{\left( a^2 \cos \varphi \partial \lambda \partial \varphi \right)}
\]

\[
\zeta = \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \delta \varphi} \frac{\partial u}{\partial \varphi} \cos \left( \frac{1}{2} \delta \varphi \right) + \frac{2u}{a} \tan \varphi \frac{\sin \left( \frac{1}{2} \delta \varphi \right)}{\delta \varphi}
\]

69
in the limit $\delta \varphi \to 0$ , $\cos \left( \frac{1}{2} \delta \varphi \right) \to 1$ and $\frac{2 \sin \left( \frac{1}{2} \delta \varphi \right)}{\frac{1}{2} \delta \varphi} \to 1$

$$\zeta = \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \frac{\partial \varphi}{\partial \varphi}} + \frac{u \tan \varphi}{a}$$

(4.75)

Equation 4.75 can be condensed to obtain

$$\zeta = \frac{1}{a \cos \varphi} \left( \frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \varphi} (u \cos \varphi) \right)$$

(4.76)

Activity 4.4

Expand equation 4.76 to obtain equation 4.75

### 4.11 Relationship Between Outflow and Divergence

While the atmosphere flow is largely rotational, it does contain a divergent component. This is a tendency of airmass to flow to a point (convergence) or flow away from a point (divergent).

We may define outflow in an analogous manner to circulation, thus

$$G = \oint V_n \, dl$$

(4.77)

Where the line integral expresses the total outflow, $V_n$ is the normal component, $dl$ is the element of the curve.
The outflow through the curve ABCDA

\[
\delta G = \sum_{i=1}^{4} V_{n_i} \delta \ell_i
\]

(4.78)

\[= V_{n_1} \delta \ell_1 + V_{n_2} \delta \ell_2 + V_{n_3} \delta \ell_3 + V_{n_4} \delta \ell_4 \]

We take \(V_{n_i}\) to be the average value of the wind component normal to the curve.

\[
V_{n_1} = -\frac{1}{2} \left( v + v + \frac{\partial v}{\partial y} \right)
\]

\[
V_{n_2} = \frac{1}{2} \left( u + \frac{\partial u}{\partial x} \delta u + u + \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right)
\]

\[
V_{n_3} = \frac{1}{2} \left( v + \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y + v + \frac{\partial v}{\partial y} \delta y \right)
\]

\[
V_{n_4} = -\frac{1}{2} \left( u + \frac{\partial u}{\partial y} \delta y + u \right)
\]

\[\delta \ell_1 = \delta x, \quad \delta \ell_2 = \delta y, \quad \delta \ell_3 = \delta x, \quad \delta \ell_4 = \delta y\]

\[
\delta G = -\frac{1}{2} \left( 2v + \frac{\partial v}{\partial x} \delta x \right) \delta x + \frac{1}{2} \left( 2u + 2 \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y \right) \delta y
\]
\[ + \frac{1}{2} \left( 2v + \frac{\partial v}{\partial x} \delta x + 2 \frac{\partial v}{\partial y} \delta y \right) \delta x - \frac{1}{2} \left( 2u + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial y} \delta y \right) \delta y \]

\[ \delta G = \frac{\partial v}{\partial x} \delta x \delta y + \frac{\partial v}{\partial y} \delta x \delta y \]

\[ \delta G = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \delta x \delta y \quad (4.79) \]

The area element is \( \delta A = \delta x \delta y \)

Dividing e.g. both side of equation 4.9 by the area we obtain

\[ \frac{\delta G}{\delta A} = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \]

\[ (4.80) \]

Or

\[ \delta G = D \delta A \]

Or

\[ D = \frac{\delta G}{\delta A} \]

The divergence may be considered as outflow per unit area.

In general

\[ G = \int V_n d\ell = \int_A D d A \quad (4.81) \]

which is Gauss theorem.
Activity 4.5

Compute the divergence between latitude circle 15°N and 20°N if the outflow is 24 m²s⁻¹.

4.12 Divergence in Spherical Co-ordinates

We will again follow the same procedure we did for deriving vorticity in spherical coordinate in section 4.8. Consider again Figure 4.14. This time we consider the velocity components that are normal to the surface thus $u'_A$, $u'_B$, $v'_C$ and $u'_D$

$$
\delta G = \sum_{i=1}^{4} V_{ni} \, d\ell_i = v'_A \, d\ell_1 + u'_B \, d\ell_2 + v'_C \, d\ell_3 + u'_D \, d\ell_4
$$

$$
= - \left( v_o - \frac{1}{2} \frac{\partial v}{\partial \phi} \delta \varphi \right) a \cos \left( \phi - \frac{1}{2} \delta \phi \right) \delta \lambda \\
+ \left( u_o + \frac{1}{2} \frac{\partial u}{\partial \lambda} \delta \lambda \right) a \delta \varphi \\
+ \left( v_o + \frac{1}{2} \frac{\partial v}{\partial \phi} \delta \phi \right) a \cos \left( \phi + \frac{1}{2} \delta \phi \right) \delta \lambda \\
- \left( u_o - \frac{1}{2} \frac{\partial u}{\partial \lambda} \delta \lambda \right) a \delta \phi
$$

Simplifying we obtain

$$
\delta G = \frac{\partial u}{\partial \lambda} a \delta \lambda \delta \varphi + \frac{\partial v}{\partial \phi} \cos \left( \frac{1}{2} \delta \phi \right) a \cos \phi \delta \lambda \delta \varphi \\
- 2v \sin \left( \frac{1}{2} \delta \phi \right) a \sin \phi \delta \lambda
$$

(4.82)

Activity 4.6
Starting from equation 4.82 and using the standard trigonometric formulae derive equation 4.83.

\[
\cos(A + B) = \cos A \cos B - \sin A \sin B \\
\cos(A - B) = \cos A \cos B + \sin A \sin B
\]

If we divide equation 4.83 by the areas \( \delta \ A = a^2 \cos \phi \ \delta \lambda \ \delta \phi \) we obtain

\[
D = \frac{\delta G}{\delta A} = \frac{\partial u}{\partial \delta \lambda} a \delta \lambda \delta \phi + \frac{\partial v}{\partial \delta \phi} \cos \left( \frac{1}{2} \delta \phi \right) a \cos \phi \ \delta \lambda \delta \phi \\
- \frac{2 \nu \sin \left( \frac{1}{2} \delta \phi \right)}{a^2 \cos \phi \ \delta \lambda \delta \phi} a \sin \phi \ \delta \lambda \ (4.84)
\]

\[
D = \frac{1}{a \cos \phi} \frac{\partial u}{\partial \delta \lambda} + \frac{1}{a} \frac{\partial v}{\partial \delta \phi} \cos \left( \frac{1}{2} \delta \phi \right) - \frac{\nu}{a} \tan \phi \ \frac{\sin \left( \frac{1}{2} \delta \phi \right)}{\frac{1}{2} \delta \phi} 
\]

In limit \( \delta \phi \to 0 \)

We have

\[
D = \frac{1}{a \cos \phi} \frac{\partial u}{\partial \delta \lambda} + \frac{1}{a} \frac{\partial v}{\partial \delta \phi} - \frac{\nu}{a} \tan \phi \ (4.85)
\]

This can be compressed to

\[
D = \frac{1}{a \cos \phi} \left( \frac{\partial u}{\partial \delta \lambda} + \frac{\partial}{\partial \delta \phi} (v \cos \phi) \right) \ (4.86)
\]

From equation 4.85 we can see that
\[ D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{v}{a} \tan \varphi \]  

(4.87)

which shows that an extra term emanate from the curvature of the earth.

Activity

- Starting from equation 4.86 work backward to obtain equation 4.85.

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LECTURE 5
CONTINUITY AND PRESSURE TENDENCY EQUATIONS

5.0 Introduction to Lecture 5

In lecture four, you learnt about circulation and examined the motions over closed paths or physical curves. By now you must have realized that the atmosphere is in ceaseless motion. The wind moves from one region to another in disregard of the political boundaries. Air being a fluid it is continuous in the free atmosphere. In this lecture you will learn about the continuity equation. As you learnt in Dynamic Meteorology I unit, the distribution of pressure determines the type of motions that are observed in the atmosphere. When the pressure drops at a particular point near the surface, it leads to the low level inflow over that region and hence upward motion. This may lead to active weather. On the other hand, when the pressure rises, it means that there is an accumulation of air mass over that region and therefore there must be an outflow of mass. A low level outflow over a given region causes a reduction of mass and is compensated for by downward flow or subsidence from the upper levels. When air mass flow from a region, it is replaced by air mass from the upper levels. Such areas are characterized by fine weather. In this regard monitoring pressure changes is key to weather forecast. You will therefore learn about the pressure tendency equation.

5.1 Objectives of Lecture 5

At the end of this lecture you should be able to:
1. Explain the meaning of continuity equation
2. Derive the continuity equation in x,y,z, t coordinates
3. Derive the continuity equation in x,y,p,t coordinates
4. State the differences between the continuity equation in x,y,z, t and x,y,p,t coordinates
5. State some of the application of the continuity equation in x,y,z,t and x,y,p,t coordinates
6. Derive the pressure tendency equation
7. Show how both the pressure tendency equation and the continuity equations are used to compute vertical velocity.
8. Discuss the use of Dines compensation law the vertical variation of the vertical velocity.

5.2 Definition of Continuity Equation

The continuity is a statement expressing the conservation of mass, or in other words there is no sources and sinks of mass anywhere in the atmosphere.
### 5.3 Derivation of Continuity Equation in x, y, z, t Coordinate.

Can you think of the implication of conservation of mass? It means that over a specified area in space over a given time, the net change in the mass of air left within the space is equal to the mass of air entering plus the mass of air leaving the space.

Consider the box whose volume is given by \( \delta v = \delta x \delta y \delta z \) and the mass is \( \rho \delta v \) if we consider a fixed volume, the change per unit time of mass is \( \frac{\partial \rho}{\partial t} \delta v \). We use the local derivative since the box is kept in the same location.

Let the velocity components at the centre of the box be \( u_o, v_o \) and \( w_o \) and density is \( \rho_o \).

The mass flux in the x-direction on face 1 the mass flux is \( \rho, u \delta y \delta z \) and on face 2 the mass flux is \( \rho_2, u_2 \delta y \delta z \). The next inflow in the x-direction is

\[
\rho, u, \delta y, \delta z = \rho_2 u_2 \delta y \delta z = \left( \rho_o u_o - \frac{1}{2} \left( \frac{\partial (\rho u)}{\partial x} \right)_o \right) \delta y \delta z
\]

\[= \left( \rho_o u_o + \frac{1}{2} \left( \frac{\partial (\rho u)}{\partial x} \right)_o \right) \delta y \delta z - \left( \frac{\partial (\rho u)}{\partial x} \right)_o \delta v \quad (5.1)
\]

On face 3 the mass flux is \( \rho_3, v_3 \delta x \delta z \)
On face 4 the mass flux is \( \rho_4, v_4 \delta x \delta z \)

The next mass inflow in the y-direction is

\[
\rho_3 v_3 \delta x \delta z - \rho_4 v_4 \delta x \delta z = \left( \rho_o v_o - \frac{1}{2} \left( \frac{\partial (\rho v)}{\partial y} \right)_o \right) \delta x \delta z
\]

\[= \rho \left( \rho_o v_o + \frac{1}{2} \left( \frac{\partial (\rho v)}{\partial y} \right)_o \right) \delta x \delta z - \left( \frac{\partial (\rho v)}{\partial y} \right)_o \delta v \quad (5.2)
\]
On face 5 the mass flux is $\rho_5, w_5 \delta x \delta y$
On face 6 the mass flux is $\rho_6, w_6 \delta x \delta y$
The net mass inflow in the z-direction.

$$\rho_5 \omega_5 \delta x \delta y - \rho_6 w_6 \delta x \delta y = \left( \rho_0 w_o - \frac{1}{2} \left( \frac{\partial (\rho w)}{\partial z} \right)_o \right) \delta x \delta y$$

$$= - \left( \frac{\partial (\rho w)}{\partial z} \right)_o \delta v$$  \hspace{1cm} (5.3)

The continuity equation is therefore

$$\frac{\partial \rho}{\partial t} = - \left( \frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v + \frac{\partial}{\partial z} \rho w \right)$$  \hspace{1cm} (5.4)

which can be written in short form as

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot (\rho \vec{v})$$ \hspace{1cm} (5.5)

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

rewriting the second term we get.

$$\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho + \rho \nabla \vec{V} = 0$$ \hspace{1cm} (5.6)

or

$$\frac{\partial \rho}{\partial t} + \rho \nabla \vec{V} = 0$$ \hspace{1cm} (5.7)

Equation 5.7 is the continuity equation in the $x, y, z, t$ coordinate. This equation shows clearly that the mass over a given location in space is due to divergence/convergence. Divergence will lead to negative mass accumulation while convergence will lead to positive mass accumulation. The limitation of the continuity equation in $x, y, z, t$ coordinate is the presence of the density in the equation which is usually difficult to determine. Nevertheless, this form of continuity equation is useful in deriving other important equations such as the pressure tendency equation.
Alternative method of derivation of continuity equation:

Consider an infinitesimal volume in space $dv$ whose surface area is $dA$

The mass flux is given by $\rho \vec{V} d\vec{A}$ where $d\vec{A}$ is the area vector.

$$d\vec{A} = \hat{n} dA$$

The total flux over the whole surface area $S$ is

$$\int S \int \rho \vec{V} d\vec{A}$$

Using Gauss theorem.

$$\int S \int \rho \vec{V} d\vec{A} = \iiint V \nabla \cdot \rho \vec{V} dv$$ \hspace{1cm} (5.8)$$

The rate of change of mass inside the volume is

$$- \iiint_v \frac{\partial \rho}{\partial t} dv$$

Therefore

$$- \iiint_v \frac{\partial \rho}{\partial t} dv = \iiint v \nabla \cdot \rho \vec{V} dv$$ \hspace{1cm} (5.9)$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$$ \hspace{1cm} (5.10)$$

5.4 Derivation of Continuity Equation in $x, y, p, t$ Coordinate

As it was indicated in course unit SMR 301, Dynamic Meteorology I, the use of pressure as a vertical coordinate simplifies the equation of motion. Let us now see what form the continuity equation takes when it is represented in the $x, y, p, t$ coordinate. We consider a ‘box’ bounded by a parallelogram with dimensions $\delta x$ and $\delta y$ and an upper and lower isobaric surface with a pressure difference $\delta p$. The box is fixed in $(x, y, p)$ – space, and the air flows through it.

We shall again use the principal that the change of mass is equal to net inflow. The mass $\delta m$ in this case is.
\[ \delta m = \frac{1}{g} \delta x \delta y \delta \rho \quad (5.10) \]

This equation is obtained by substituting for \( \rho \delta z \) in \( \rho \delta x \delta y \delta z \) by \( \frac{\delta \rho}{g} \) and notice \( \delta x, \delta y \), \( \delta \rho \) are constants. The net mass inflow in the x-direction is

\[ u, \delta y \frac{\delta \rho}{g} - u_2 \delta y \frac{\delta \rho}{g} \quad (5.11) \]

The net mass inflow in the y-direction

\[ V_3 \delta x \frac{\delta \rho}{g} - V_4 \delta x \frac{\delta \rho}{g} \quad (5.12) \]

The net mass inflow in the \( \rho \)-direction

\[ \frac{\omega_s}{g} \delta x \delta y - \frac{\omega_s}{g} \delta x \delta y \quad (5.13) \]

Expanding equation 5.11 we get

\[ \left( u_o - \frac{1}{2} \frac{\partial u}{\partial x} \right) \delta x \delta y \frac{\delta \rho}{g} - \left( u_o + \frac{1}{2} \frac{\partial u}{\partial x} \right) \delta x \delta y \frac{\delta \rho}{g} \quad (5.14) \]

From equation 5.12 we obtain:

\[ \left( V_o - \frac{1}{2} \frac{\partial v}{\partial y} \right) \delta y \delta x \frac{\delta \rho}{g} - \left( V_o + \frac{1}{2} \frac{\partial v}{\partial y} \right) \delta y \delta x \frac{\delta \rho}{g} \]

\[ = - \frac{\partial v}{\partial y} \delta x \delta y \delta \frac{\delta \rho}{g} \quad (5.15) \]

From equation 5.13 we obtain

\[ \left( \omega_o - \frac{1}{2} \frac{\partial \omega}{\partial \rho} \right) \delta \rho \delta y \frac{\delta \rho}{g} - \left( \omega_o + \frac{1}{2} \frac{\partial \omega}{\partial \rho} \right) \delta \rho \delta y \frac{\delta \rho}{g} \]

\[ = - \frac{\partial \omega}{\partial \rho} \delta x \delta y \delta \frac{\delta \rho}{g} \quad (5.16) \]
The change per unit time of mass is \( \frac{\partial}{\partial t} \left( \frac{1}{g} \delta x \delta y \delta \rho \right) \)

Which is zero since \( \delta x, \delta y \) and \( \delta \rho \) are constant

Hence the sum of 5.14, 5.15 and 5.16 is equal to zero.

Hence

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial \rho} = 0 \quad (5.17)
\]

which can be written in short form as

\[
\nabla \cdot \vec{V} = 0 \quad (5.18)
\]

When you compare equation 5.18 and 5.17 we see that equation 5.18 is much simpler.
The other advantage of the continuity equation is that it does not contain density.

5.5 Continuity Equation for Incompressible Fluid

Starting from the continuity equation we have already derived.

\[
\frac{d\rho}{dt} + \rho \nabla \cdot \vec{V} = 0
\]

Compressibility implies that

\[
\frac{d\rho}{dt} = 0
\]

hence \( \nabla \cdot \vec{V} = 0 \) \quad (5.19)

which is similar to the continuity equation in \( x, y, p, t \) coordinate.

5.6 Computation of Vertical Motion Using Continuity Equation

We need to mention here that the vertical component of velocity is not usually measured, it has to be computed. One method of computing it is to use the continuity equation in the form given in equation 5.19 for the \( x, y, z, t \) coordinates and equation 5.18 for the \( x, y, p, t \) coordinate.
We start with the \( x, y, z, t \) coordinates.

\[
\nabla \cdot \vec{V} = 0
\]

Or

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = - \frac{\partial \omega}{\partial z}
\]

where \( D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \) is the horizontal divergence

\[
\frac{\partial \omega}{\partial z} = -D
\]  \hspace{1cm} (5.20)

If we know the vertical velocity at some level \( z_o \), we can integrate equation 5.20 to find the vertical velocity at some level \( x \).

\[
\int_{z_o}^{z} \frac{\partial \omega}{\partial z} = \int_{z_o}^{z} Ddz
\]

\[
\omega (z) = \omega (z_o) - \int_{z_o}^{z} Ddz
\]  \hspace{1cm} (5.21)

From 5.21 we note that if there is no horizontal divergence, there will be no change in the vertical motion.

Next we examine the vertical motion in the \( x, y, p, t \) coordinate. Notice that when \( p \) is used as a vertical coordinate the vertical velocity is given as

\[
\omega = \frac{dp}{dt}
\]

From equation 5.17 we have

\[
\frac{\partial \omega}{\partial p} = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]

\[
\frac{\partial \omega}{\partial p} = -D
\]  \hspace{1cm} (5.22)
Again if we know $\omega$ at some level $z_o$, we can integrate 5.22 to get the vertical velocity at $z$.

$$\int_{z_o}^{z} \partial \omega = -\int_{z_o}^{z} D \partial \rho$$

$$\omega(z) = \omega(z_o) - \int_{z_o}^{z} D \partial \rho$$  \hspace{1cm} (5.23)

**Activity 5.1**

Calculate the vertical velocity at 700mb and 500mb, if the pressure at 850mb is decreasing by 8mb per day and the vertical variation of divergence is

$$D = D_o \sin 2\pi \left( \frac{\rho - 500}{1000} \right)$$

where $D_o = 4 \times 10^{-6} \, s^{-1}$

---

### 5.6 Pressure Tendency Equation

If you examine the surface weather charts, you will find that one of the parameters that is plotted on it is the pressure tendency. This parameter is very important in determining the changes in weather. We will now show that pressure tendency is related to horizontal mass divergence.

Starting from the Hydrostotic Equation

$$\frac{\partial \rho}{\partial z} = -\rho g$$  \hspace{1cm} (5.24)

We integrate equation 5.54 with respect to $z$, from some level $z$ to the top of the atmosphere at $z = 0$

$$\int_{z}^{\infty} \partial \rho = -\int_{z}^{\infty} \rho g dz$$  \hspace{1cm} (5.25)

which yields
\[-P = \int_{z}^{\infty} \rho \, g \, d \, z\]

or

\[P = \int_{z}^{\infty} \rho \, g \, d \, z\]  \hspace{1cm} (5.26)

Differentiating (partial) equation 5.26 with respect to time, we obtain

\[\frac{\partial \rho}{\partial t} = g \int_{z}^{\infty} \frac{\partial \rho}{\partial t} \, d \, z\] \hspace{1cm} (5.27)

We substitute right hand side of equation 5.27 using the continuity equation.

\[\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \, \bar{V}\]

\[\frac{\partial \rho}{\partial t} = -g \int_{z}^{\infty} \nabla \cdot \rho \, \bar{V} \, d \, z\] \hspace{1cm} (5.28)

Expanding the right hand side we obtain

\[\frac{\partial \rho}{\partial t} = -g \int_{z}^{\infty} \nabla \cdot \rho \, \bar{V} \, d \, z - g \int_{z}^{\infty} \frac{\partial}{\partial z}(\rho \, \omega) \, d \, z\] \hspace{1cm} (5.29)

which simplifies to

\[\frac{\partial \rho}{\partial t} = -g \int_{z}^{\infty} \nabla \cdot \rho \, \bar{V} \, d \, z + g \rho \, \omega\] \hspace{1cm} (5.29)

since \(\omega_{\infty} = 0\)

From equation 5.29, we see that the pressure tendency is due to both vertical motion and horizontal divergence.

If \(z\) is at the surface, and we consider a case where it is flat. Then the vertical velocity will be zero, in which case the contribution will only be from the vertical integration of divergence.

Thus

\[\frac{\partial \rho}{\partial t} = -g \int_{0}^{\infty} \nabla \cdot \rho \, \bar{V} \, d \, z\] \hspace{1cm} (5.30)
Scale analysis of equation 5.28 show that the left hand term is an order of magnitude lower compared to the terms on the right hand side.

Thus

\[ \int_z^\infty \nabla \cdot \rho \vec{V}_h \, dz = \rho \omega \quad (5.31) \]

Activity 5.2

Using the characteristic values \( \Delta \rho \sim 10^3 \text{kgm}^{-1} \text{s}^{-2}, \rho = 1 \text{kg m}^3, \Delta t \sim 10^5 \text{s}, g \sim 10 \text{m s}^{-2}, \omega \sim 10^{-2}, \nabla \sim 10^{-6}, |\vec{V}| \sim 10 \text{ms}^{-1} \) show that the terms on the right hand side of equation 5.29 are an order of magnitude higher than the term on the left hand side.

5.7 Dines Compensation Law

Observational studies show that over the whole depth of the atmosphere, there is divergence in some layers and convergence in skies so that the total sum of divergence in the vertical is almost equal to zero. This led Dine to suggest that Divergence (convergence) in the lower levels is compensated for by convergence (divergence) in upper levels.

in tropics

\[ \omega = \frac{1}{\rho} \int_z^\infty \nabla \cdot \rho \vec{V} \, dz \quad (5.32) \]

From equation 5.32, it is clear that at the level of non-divergence, the vertical motion would be maximum.

5.8 Other Methods of Computing Vertical Motion

Vertical motion can be computed from the first law of thermodynamic.
\[
\frac{dq}{dt} = C_\rho \frac{dT}{dt} - \alpha \frac{d\rho}{dt}
\]  
(5.33)

from which we can obtain

\[
\omega = \frac{C_\rho \left( \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right) - \dot{q}}{\left( \frac{1}{\rho} - C_\rho \frac{\partial T}{\partial \rho} \right)}
\]  
(5.34)

in \(x, y, p, t\) coordinate

and

\[
\omega = \left( q \left/ C_\rho \left( \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T \right) \right. \right. \left. \left/ \rho \left( \frac{\partial T}{\partial \rho} \right) \right. \) \right.
\]  
(5.35)

in \(x, y, z, t\) coordinate where \(\dot{q} = \frac{dq}{dt}\)

**Activity 5.3**

Starting from equation 5.33 derive  
(a) Equation 5.34  
(b) Equation 5.35

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LECTURE 6
THE ATMOSPHERIC OSCILLATION

6.0 Introduction to Lecture six
An oscillation refers to any process that repeats itself at regular intervals. An example of oscillation is a pendulum swinging forth and back. The position of the sun can be said to oscillate from the tropical of Capricorn to the tropical of Cancer and back. There are many natural oscillations in the atmosphere. Some of them cause changes in the weather but others do not. In Lecture 7 you will learn about specific oscillations in the atmosphere. But in lecture you will learn about the general characteristics of oscillatory motions.

6.1 Lecture Objectives

At the end of this lecture, you should be able to:
1. Describe the general characteristics of a harmonic and periodic functions.
2. State the difference between free and damped motions
3. Solve the differential equations that represent harmonic functions
4. Use Fourier decomposition to transform a function to a periodic function.
5. Give examples of atmospheric oscillation.

6.2 Periodic Functions
A function that is oscillatory with respect to time is referred to as a periodic function.
A function of the form
\[ F(t) = A \sin(\alpha t) \]
……………………………………………………………………………….. 6.1
The minimum value of $\sin(\alpha t)$ is 1 while the minimum is -1.

Now suppose $\alpha = 1$, then $\sin(\alpha t)$ will have maximum value at $t = \frac{\pi}{2} + 2\pi n$ where $n = 0, 1, 2, \ldots$. From which we see that the maximum occurs at interval of $2\pi$. This time interval is referred to as the **period** of the function.

Next, let us look at the case when $\alpha = 2$. The first maximum will occur at $2t = \frac{\pi}{2}$ or $t = \frac{\pi}{4}$ and the second maximum will occur at $2t = \frac{\pi}{2} + 2\pi$ or $t = \frac{\pi}{4} + \pi$. The time interval in this case is $\pi$. By increase $\alpha$ it can be seen that the period decreases.

**Activity 6.1**

1. Consider the equation $G(t) = 12 \sin(\omega t)$. Find the time at which the first maximum will occur if we are starting at zero time for $\omega = 5, 4, 3, 2, 1, \frac{1}{8}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$. For each case compute the period.

2. Consider the equation $H(t) = 12 \cos(\beta t)$. Find the time at which the first maximum will occur if we are starting at zero time for $\beta = 5, 4, 3, 2, 1, \frac{1}{8}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$. For each case compute the period.

3. Compare the results you have obtained from question 1 and 2. What are the similarities and difference about the two functions?
6.2.1 Simple Harmonic Motion

A simple harmonic motion is a motion of a particle whose acceleration is always:

a) directed towards a fixed point

b) directly proportional to its distance from the fixed point

If \( \ddot{a} \) is the acceleration and \( \vec{r} \) the displacement. Then for simple harmonic motion

\[
\ddot{a} = -\alpha \vec{r} \quad \text{................................................................. 6.2}
\]

Where \( \alpha \) is a constant.

Let us consider the motion in the x-direction

The acceleration \( a \), is given by

\[
a = \frac{du}{dt} = \frac{dx}{dt} \frac{du}{dx} = u \frac{du}{dx} \quad \text{..................... 6.3}
\]

From equation 6.2 by definition

\[
a = -\alpha x \quad \text{................................................................. 6.4}
\]

From 6.3 and 6.4

\[
u \frac{du}{dx} = -\alpha x \quad \text{................................................................. 6.5}
\]

Integrating 6.5 we obtain

\[
\frac{u^2}{2} = -\frac{\alpha x^2}{2} + c \quad \text{................................................................. 6.6}
\]

For simple harmonic motion the speed is equal zero when the displacement \( x \), is maximum, \( x=A \) Where A is the amplitude

Thus \( c = \frac{\alpha A^2}{2} \quad \text{................................................................. 6.7} \)

From 6.6 and 6.7 we obtain

\[
u = \sqrt{\alpha} \sqrt{A^2 - x^2} \\
\frac{dx}{dt} = \sqrt{\alpha} \sqrt{A^2 - x^2} \quad \text{................................................................. 6.8}
\]
Integrating

\[ \frac{1}{\sqrt{\alpha}} \int \frac{dx}{\sqrt{A^2 - x^2}} = \int dt \] ................................. 6.9

Solving equation 6.9 we obtain

\[ \frac{1}{\sqrt{\alpha}} \sin^{-1}\left(\frac{x}{A}\right) = t + c \]

When t=0, then x=0 so c=0

Therefore \[ \frac{1}{\sqrt{\alpha}} \sin^{-1}\left(\frac{x}{A}\right) = t \] ................................. 6.10

\[ x = A\sin\left(\sqrt{\alpha} t\right) \]

When t increases to \( t + \frac{2\pi}{\sqrt{\alpha}} \), \( x = A\sin(\sqrt{\alpha} t + 2\pi) = A\sin\sqrt{\alpha} t \), which is the same displacement as at the value t. Hence the period of the motion is \( \frac{2\pi}{\sqrt{\alpha}} \)

6.2.2 Application of Calculus in finding Equations of Simple Harmonic Motion

The solution of a second order differential equation can be solve by reducing it to first order and solving twice

For example \( \frac{d^2x}{dt^2} + x = 0 \) ........................................6.11

Let

\[ \frac{dx}{dt} = q \text{ then } \frac{d^2x}{dt^2} = \frac{dq}{dt} = \frac{dq}{dx} \frac{dx}{dt} = q \frac{dq}{dx} \] ................................. 6.12

Therefore

\[ q \frac{dq}{dx} + x = 0 \]

or \[ qdq + xdx = 0 \] ................................................. 6.13

\[ \int \frac{dx}{\sqrt{A^2 - x^2}} = \int dt \]
From 6.12
\[
\frac{q^2}{2} + \frac{x^2}{2} = \frac{C_1}{2}
\]…………………………….                                6.14

Or
\[
q = \pm \sqrt{\frac{C_1}{2} - x^2}
\]
but \( q = \frac{dx}{dt} \)
Therefore
\[
\frac{dx}{dt} = \pm \sqrt{\frac{C_1}{2} - x^2}
\]
or
\[
\frac{dx}{\sqrt{C_1 - x^2}} = \pm dt
\]

The solution is
\[
\sin^{-1}\left(\frac{x}{C_1}\right) = \pm(t + c_2).
\]
or
\[
x = c_1 \sin\left(\pm (t + c_2)\right)
\]

\[
x = \pm c_1 \sin\left(t + C_c\right)
\]

6.2.3. The phase shift

Suppose you have three functions, namely ;

\[
F(t) = 10\sin(3t)
\] 6.2

\[
F_2(t) = 10\sin(3t + \frac{\pi}{4})
\] 6.3

\[
F_3(t) = 10\sin(3t - \frac{\pi}{4})
\] 6.4

What do you think is the basic difference between these functions? Work out the period of each function. You will discover that the three functions have the same period so what is the difference. Now find the time at which the first maximum occurs for each function.
Did you find that the time of the maximum of $F_3(t)$ later than $F(t)$ while $F_3(t)$. The shift in the time of the occurrence of the maximum or minimum is referred to as the Phase shift.

**Activity 6.2**

1. Plot the function given in equations 6.2, 6.3 and 6.4 on the same graph for $0 \leq t \leq 10$. Comment on the results.
2. Plot the following functions on the same graph for the range of values $0 \leq t \leq 10$

$$q_3(t) = 8\cos(t - \pi)$$
$$q_2(t) = 8\sin(t - \frac{\pi}{2})$$
$$q_1(t) = 8\sin(t)$$

What do you notice about the graph of $q_3$ and $q_1$.

You should have noted in activity 6.2 question 2 that the graph for $q_3$ and $q_1$ are in the opposite direction. When such a feature occurs we say that the two graphs are out of phase. This always occurs when the phase shift is 180.

In general if you have a function $W(t) = A\cos(\beta t + \phi)$

The period is given by $T = \frac{2\pi}{\beta}$ the reciprocal of this value $\frac{1}{T}$ is the frequency

The frequency is the number of maximum (or minimum) values of $W(t)$ that occur within the range of time, $-\phi \leq t \leq 2\pi - \phi$. 

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6.2.4 Wavelength and Wave Number

Functions can also fluctuate at regular intervals with respect to space. Now consider the function
\[ Y(x) = A \cos(kx) \] ................................. 6.5

If the x values are starting from zero the first minimum of the function \( Y(x) \) will occur at \( x = \frac{\pi}{k} \). The second minimum will occur at \( \frac{\pi}{k} + \frac{2\pi}{k} \). Thus the interval between two successive minima or maxima is \( \frac{2\pi}{k} \).

If \( k = 1 \), then there will be one minima in the range \( 0 \leq x \leq 2\pi \). It can be seen that as the value of \( k \) increases the number of waves within the interval \( 0 \leq x \leq 2\pi \) increases. \( k \) is therefore referred to as the wave number.

Mathematically the wave number is defined as
\[ k = \frac{2\pi}{\lambda} \] where \( \lambda \) is the wavelength. The wavelength is the distance between two successive minima or maxima.

6.2.5 The Phase Speed

Functions sometimes depend on both time and space. Consider the function
\[ R(x, t) = A \cos k(x - ct) \] ................................................................. 6.6

What is the period of this function?

If we rewrite this function as
\[ R(x, t) = A \cos(-kct + kx) \] or
\[ R(x, t) = A \cos(\gamma t + kx) \] where \( \gamma = -kc \)

Then the period is given by \( \frac{2\pi}{|\gamma|} \).
Activity 6.3

1. Given the function \( h(x,t) = 10 \cos k(x - ct) \), On the same graph plot the function \( h(x,t) \) for time interval \( 0 \leq t \leq 500 \) at \( x=0 \) and \( x=350 \)m if the wave length is 1500m and \( c=10 \text{ms}^{-1} \)
   (a) Find the period of this wave
   (b) What is the wave number?
2. Plot the function given in Q1 at \( t=0 \) and \( t=45 \)sec for the range of distance given by \( 0 \leq x \leq 4000 \)

6.2.6 Fourier Decomposition of Periodic Functions

In the atmosphere we have waves of different wavelengths that oscillate at different frequency. These waves may interact so that what we observe is a combination of various waves. The resultant wave is usually quasi period. For example suppose we have \( n \) waves, namely: \( y_1(x,t), y_2(x,t), \ldots, y_n(x,t) \).

The resultant wave will be given by
\[
y(x,t) = y_1(x,t) + y_2(x,t) + \ldots + y_n(x,t) \]

or
\[
y_i(x,t) = a_i \cos(k_i(x - c_it))
\]
\[
y(x,t) = \sum_{i=1}^{n} a_i \cos(k_i(x - c_it)) \]

If \( y(x,t) \) is known then we can determine \( \nu_i = k_i c_i \) the frequency of constituency waves. The process is referred to Fourier decomposition or simply Fourier Transform.

The transformation of a function \( f(x) \) is only possible if the function is defined within \( (0<x2\pi) \).
For example \[ f(x) = \begin{cases} 1 & \text{for } 0 < x < \pi \\ 2 & \text{for } \pi < x < 2\pi \end{cases} \] \hspace{1cm} 6.9

The transformation is

\[ f(x) = \sum_{K=0}^{K=n} (a_k \cos kx + b_k \sin bx) \] \hspace{1cm} 6.10

The task is to determine the coefficient \( a_k \) and \( b_k \).

These coefficients are obtained by using orthogonality property of sines and cosines.

\[ \int_0^{2\pi} \sin(jx) \sin(kx) dx = \begin{cases} 0 & \text{if } j \neq k \\ \pi & \text{if } j = k \neq 0 \end{cases} \] \hspace{1cm} 6.11

\[ \int_0^{2\pi} \sin(jx) \cos(kx) = 0 \] \hspace{1cm} 6.12

\[ \int_0^{2\pi} \cos(jx) \cos(kx) dx = \begin{cases} 0 & \text{if } j \neq k \\ \pi & \text{if } j = k \neq 0 \\ 2\pi & \text{if } j = k = 0 \end{cases} \] \hspace{1cm} 6.13

\[ a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(kx) dx, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(kx) dx \] \hspace{1cm} 6.14

The Fourier Spectrum

The Fourier transform is used to detect the hidden periodicity. It is also used to transform the equation if motion into spectra form. If for example the function \( f(\lambda) \) where \( \lambda \) is the longitude \((0 < \lambda, 2\pi)\) is to be expanded we will have the following.

\[ f(\lambda) = \sum_{k=-\infty}^{\infty} F(k)e^{-ik\lambda} \] \hspace{1cm} 6.15

Where the complex coefficients \( F(k) \) are given by
\[ F(k) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\lambda) e^{ik\lambda} \, dx \quad (k=0,1,2\ldots) \quad \cdots \cdots \quad 6.16 \]

K is the wave number of the waves around the latitude circle.

Equation 6.15 and 6.16 constitute a Fourier transform pair. \( F(k) \) which is the representation of \( f(\lambda) \) in the wave number domain, is called the complex spectral function of \( f(\lambda) \).

The expression in 6.16 can be written in the form

\[ F(k) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\lambda) \cos(k\lambda) \, d\lambda + \frac{i}{2\pi} \int_{0}^{2\pi} f(\lambda) \sin(k\lambda) \, d\lambda \quad \cdots \cdots \quad 6.17 \]

or \( F(k) = F_1(k) + F_2(k) \)

Where \( F_1(k) \) and \( F_2(k) \) are the real and imaginary parts of \( F(k) \). It can easily be shown that

\[ F_1(-k) = F_1(k) \quad \cdots \cdots \quad 6.18 \]
\[ F_2(-k) = -F_2(k) \]

Thus, the real part is an even function of \( k \) and the imaginary part the odd function of \( k \). Furthermore

\[ F(-k) = F_1(k) - iF_2(k) = \hat{F}(k) \quad \cdots \cdots \quad 6.19 \]

If we denote the complex conjugate \( F(k) \) by \( \hat{F}(k) \). The spectral function \( F(k) \) can also be written in the exponential form

\[ F(k) = |F(k)| e^{ik\theta_k} \quad \cdots \cdots \quad 6.20 \]

Where the amplitude \( |F(k)| \) and the phase angle \( \theta_k \) are given in terms of the real and imaginary parts by the expression

\[ |F(k)| = \left( F_1^2 + F_2^2 \right)^{1/2} \quad \cdots \cdots \quad 6.21 \]

And
\[ \theta'_k = \frac{1}{k} \tan^{-1} \left\{ \frac{F_2(k)}{F_1(K)} \right\} \]  \hspace{1cm} 6.22

Substituting 6.20 into 6.15 we get a new form for the Fourier expansion of \( f(\lambda) \) in the wave domain

\[ f(\lambda) = \sum_{k=-\infty}^{\infty} |F(k)| e^{-ik(\lambda - \theta'_k)} \]  \hspace{1cm} 6.23

Which simply be written as

\[ f(\lambda) = F(0) + 2 \sum_{k=1}^{\infty} |F(k)| \cos k(\lambda - \theta'_k) \]  \hspace{1cm} 6.24

Equation 6.24 could easily be derived from equation 6.15 by rewriting it as

\[ f(\lambda) = F1(0) + \sum_{k=1}^{\infty} \left\{ F(k)e^{-ik\lambda} + \hat{F}(k)e^{ik\lambda} \right\} \]  \hspace{1cm} 6.25

Or

\[ f(\lambda) = F(0) + 2 \sum_{k=1}^{\infty} \left\{ F_1(k) \cos k\lambda + F_2(k) \sin k\lambda \right\} \]  \hspace{1cm} 6.26

Since \( F_2(0) = 0 \)

If we let

\[ a_k = 2F_1(k) \]
\[ b_k = 2F_2(k) \]

We obtain
\[ f(\lambda) = \frac{a_o}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos k\lambda + b_k \sin k\lambda \right\} \]

\[ f(\lambda) = \sum_{k=0}^{\infty} c_k \cos k(\lambda - \theta_k) \]

where \( c_o = \frac{a_o}{2} \)

\[ c_k = \left( a_k^2 + b_k^2 \right)^{1/2} \] ... \( fork = 1,2,\ldots,\infty \)

and

\[ \theta_k = \frac{1}{k} \tan^{-1} \left( \frac{b_k}{a_k} \right) \]

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LECTURE 7

ATMOSPHERIC WAVES

7.0 Introduction to lecture seven

In the previous lecture we examined the general characteristics of waves. In this lecture we shall mainly concentrate on the meteorological waves. Meteorological waves are waves that influence the weather. The equations of motion support all types of waves, we require a method that can isolate them. This lecture is devoted to the procedure of obtaining various meteorological waves from the equations governing the atmospheric motions.

7.1 Lecture Objectives

At the end of this lecture you should be able to:

1. State the perturbation theory
2. Use the perturbation method to solve a complex differential equation
3. Derive the phase speed for sound waves, gravity waves, Rossby waves and mixed Rossby gravity waves
4. Discuss the importance of meteorological waves in weather analysis and prediction

7.2 The Perturbation Method of Solving Differential Equations

The general meteorological equations are too complex for one to obtain an analytical solution as we were able to get for simple differential equations in lesson 6. In this lecture we will learn about the perturbation method which is used to simplify the equation so that they are solved analytically. In lecture 9, you will learn about the numerical methods that are used to solve the equations governing the atmospheric motions.

The main purpose of the perturbation method is to reduce the non-linear equations to a set of linear equations. This is done under simplifying assumptions.

The general aspects of the perturbation method are as follows:

1. The set of equations governing the physical problems at hand correspond to the basic state of the atmosphere. This state is denoted by symbols: $\bar{u}, \bar{v}, \bar{w}, \bar{\rho}, \bar{p}, \bar{T}$. 
2. We consider a new state consisting of the basic state plus an additional field, called the perturbation field. We denote this field by primed symbols we get; 
\[ \bar{u} + u', \bar{v} + v', \bar{w} + w', \bar{p} + p', \bar{T} + T' \]. Now we seek to solve the equations to obtain the total field e.g. \( \bar{u} + u' \) etc.

3. After we have substituted the total field \( (\bar{u} + u', etc) \) into the equations, we make the major assumption in the perturbation method that the perturbation in small so that we neglect all quantities which are of second or higher order in the perturbation quantities. This assumption reduces the non-linear equations to a set of linear partial differential equations, which in certain cases can be solved mathematically.

4. Solutions to the linear equations, describing the behaviour of the perturbation fields, are sought in terms of the wave type solutions. Thus determining the solution for the wave speed \( (C = Cr + iCi) \).

5. From the solution we may determine if the basic state is unstable \( (C_i > 0) \), neutral \( (C_i = 0) \) and stable \( (C_i < 0) \). We main in addition determine the speed \( (C_i) \) of wave and the structure of the perturbation wave.

7.3 The Equation of Sound Waves

Sound waves are propagated through the atmosphere that is why we hear sounds. The sound is propagated by the movement of the air molecules. We will therefore derive the equation which represents the speed of sound. In order to get the sound waves in a pure form we shall consider a particularly simple case.

(a) We assume the basic state consist of an airmass in which we may assume that \( \bar{\alpha} \) is a constant (homogenous atmosphere), and in which there is no motion, i.e. \( \bar{u} = \bar{v} = \bar{w} = 0 \). The basic state is therefore described by the equation

\[ \bar{P} = \text{constant} \]  

which implies

\[ \frac{\partial \bar{P}}{\partial x}, \frac{\partial \bar{P}}{\partial y}, \frac{\partial \bar{P}}{\partial z} = 0 \]  

(b) The total field is: \( u = u', v = v', w = w', p = \bar{p} + p' \) and \( \alpha = \bar{\alpha} + \alpha' \)

(c) Since the sound waves moves very fast we can neglect
The rotation of the earth, thus $f = 0$

The adiabatic heating. Sound moves very fast that the change in heating or cooling from external can be neglected.

Friction is also neglected.

The equation describing the above situation are:

$$\frac{\partial u}{\partial t} = -\alpha \frac{\partial \rho}{\partial x}$$  \hspace{1cm} (7.3)

$$\frac{\partial v}{\partial t} = -\alpha \frac{\partial \rho}{\partial y}$$  \hspace{1cm} (7.4)

$$\frac{\partial w}{\partial t} = -\alpha \frac{\partial \rho}{\partial z}$$  \hspace{1cm} (7.5)

$$\frac{\partial \rho}{\partial t} = -\rho \nabla \cdot \vec{V} \quad \text{where} \quad \gamma = \frac{c}{c_p}$$  \hspace{1cm} (7.6)

Expanding equation 7.3 we obtain

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\alpha \frac{\partial \rho}{\partial x}$$  \hspace{1cm} (7.7)

Substituting with the total field

$$\frac{\partial u'}{\partial t} + u' \frac{\partial u'}{\partial x} + v' \frac{\partial u'}{\partial y} + w' \frac{\partial u'}{\partial z} = -\alpha \frac{\partial \overline{p}}{\partial x} - \alpha \frac{\partial \rho'}{\partial x} - \alpha' \frac{\partial \overline{p}}{\partial x} - \alpha' \frac{\partial \rho'}{\partial x}$$  \hspace{1cm} (7.8)

Applying the following condition

(i) The product of perturbation are negligible

(ii) $\frac{\partial \overline{p}}{\partial x} = 0$

Equation 7.8 reduces to

$$\frac{\partial u'}{\partial t} = -\alpha \frac{\partial \rho'}{\partial x}$$  \hspace{1cm} (7.9)

Following the same procedure we obtain
\[
\frac{\partial v'}{\partial t} = -\alpha \frac{\partial p'}{\partial y} 
\] (7.10)

\[
\frac{\partial w'}{\partial t} = -\alpha \frac{\partial p'}{\partial z} 
\] (7.11)

\[
\frac{\partial p'}{\partial t} = -\frac{\varphi p}{\rho_0} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right) 
\] (7.2)

Activity 7.1

Starting from the thermodynamic equation \( dh = C_v dT + \rho d\alpha \) and applying the assumption applied to the sound waves obtain equation 7.6.

We eliminate \( u', v' \) and \( w' \) from the equation 7.12 by differentiating this equation and substituting with equation and substituting with equation 7.9 to 7.11.

Thus

\[
\frac{\partial^2 p'}{\partial t^2} = -\frac{\partial}{\partial t} \left[ \frac{\varphi p}{\rho_0} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right) \right]
\]

which yields

\[
\frac{\partial^2 p'}{\partial t^2} = -\frac{\varphi p}{\rho_0} \left( \frac{\partial}{\partial x} \left( \frac{\partial u'}{\partial t} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v'}{\partial t} \right) + \frac{\partial}{\partial z} \left( \frac{\partial w'}{\partial t} \right) \right) 
\] (7.13)

since \( \frac{\partial \rho}{\partial t} = 0 \)

On substituting we obtaining

\[
\frac{\partial^2 p'}{\partial t^2} = -\frac{\varphi p}{\rho_0} \left[ \frac{\partial}{\partial x} \left( -\alpha \frac{\partial p'}{\partial x} \right) + \frac{\partial}{\partial y} \left( -\alpha \frac{\partial p'}{\partial y} \right) + \frac{\partial}{\partial z} \left( -\alpha \frac{\partial p'}{\partial z} \right) \right]
\]

Thus
\[
\frac{\partial^2 p'}{\partial t^2} = \phi \alpha \ p \nabla^2 \ p'
\]  
(7.14)

Let \( \mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z \)

\[
\frac{\partial p'}{\partial r} \hat{r} = \frac{\partial p'}{\partial x} \hat{i} + \frac{\partial p'}{\partial y} \hat{j} + \frac{\partial p'}{\partial z} \hat{k}
\]

\[
\frac{\partial}{\partial r} \hat{r} \cdot \frac{\partial p'}{\partial r} = \left( \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left( \frac{\partial p'}{\partial x} \hat{i} + \frac{\partial p'}{\partial y} \hat{j} + \frac{\partial p'}{\partial z} \hat{k} \right)
\]

\[
\frac{\partial^2 p}{\partial r^2} = \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2}
\]

Hence

\[
\frac{\partial^2 p'}{\partial t^2} = \phi \alpha \ p \frac{\partial^2 p'}{\partial r^2}
\]  
(7.15)

Assuming that

\[
p' = \hat{p} \cos k(r - ct)
\]  
(7.16)

We differentiate equation 7.16 twice w.r.t. line

\[
\frac{\partial p'}{\partial t} = \hat{p} \ k \ c \ \sin k(r - ct)
\]

\[
\frac{\partial^2 p'}{\partial t^2} = - \hat{p} \ k^2 \ c^2 \ \cos k(r - ct)
\]

Thus

\[
\frac{\partial^2 p'}{\partial t^2} = - k^2 \ c^2 \ p'
\]  
(7.17)

Differentiating equation 7.16 to r.

\[
\frac{\partial p'}{\partial r} = - \hat{p} \ k \ \sin k(r - ct)
\]
\[
\frac{\partial^2 p'}{\partial r^2} = -\hat{p} k^2 \cos k (r - ct)
\]

\[
\frac{\partial^2 p'}{\partial r^2} = -k^2 p'
\]  
(7.18)

Substituting equation 7.17 and 7.18 into 7.15 we obtain

\[
-k^2 c^2 p' = -\varphi \bar{\alpha} \bar{p} k^2 p'
\]

which leads to

\[
C = \pm \sqrt{\varphi \bar{\alpha} \bar{p}}
\]  
(7.19)

Using the equation of state

\[
\bar{p} \bar{\alpha} = R \bar{T}
\]

\[
\bar{\alpha} = \frac{R \bar{T}}{\bar{p}}
\]  
(7.20)

Substituting 7.20 into 7.19 we obtain

\[
C = \pm \sqrt{R \bar{T}}
\]

For \( C_p = 1004 \text{ m}^2 \text{s}^{-2} \text{deg}^{-1}, \ C_v = 717 \text{ m}^2 \text{s}^{-2} \text{deg}^{-1}, \ R = 287 \text{ m}^2 \text{s}^{-2} \text{deg}^{-1} \) and \( \bar{T} = 273^0 k \) we find \( C = 331 \text{ ms}^{-1} \) which may be considered as the speed of sound in an isothermal atmosphere. The sound waves propagate in the atmosphere through a service of compressions and expansions, and they will therefore only propagate if the medium is compressible.

### 7.4 The Equation of Gravity Waves

There are two types of gravity waves

i. Internal gravity waves

ii. External gravity waves
The external gravity waves occur when the whole atmosphere is disturbed. The internal gravity waves on the other hand occur when the disturbance is internally set up, for example the flow impinging on a mountain, or due to bouyance force resulting to form local heating of the atmosphere from below.

### 7.4.1 Equation for Internal Gravity Waves

For simplicity we consider motion in the x-z plane.

We assume the following

We assume the mean pressure is dependent only on the vertical, $\bar{P} = \bar{P}(z)$

Neglect the mean horizontal motion $\bar{u} = 0$

In the basic state that the atmosphere is in hydrostatic balance.

$$\bar{\alpha} \frac{\partial \bar{p}}{\partial z} + g = 0$$

$\bar{\alpha}$ is constant

$\bar{w} = 0$

$$u = u'$$

$$w = w'$$

$$p = \bar{p} + p', \quad \bar{p} = \bar{p}(z)$$

$$\alpha = \bar{\alpha} + \alpha', \quad \bar{\alpha} = \bar{\alpha}(z)$$

Friction and rotation of the earth are neglected equate of state.

$$\bar{p} \quad \bar{\alpha} = R \bar{T}$$

The basic equations are:

$$\frac{du}{dt} + \alpha \frac{\partial p}{\partial x} = 0 \quad \text{(horizontal motion)}$$

(7.21)
Expanding equation 7.2 to 7.24 and applying the assumption stated above and after manipulation we obtain

\[
\frac{\partial u'}{\partial t} + \alpha' \frac{\partial p'}{\partial x} = 0
\]  
(7.26)

\[
\frac{\partial w'}{\partial t} + \alpha' \frac{\partial p'}{\partial z} - \frac{\alpha'}{\alpha} g = 0
\]  
(7.27)

\[
\varphi \frac{\partial p'}{\partial t} + \varphi w' \frac{\partial \alpha}{\partial z} + \alpha \frac{\partial p'}{\partial t} - gw' = 0
\]  
(7.28)

\[
\frac{\partial \alpha'}{\partial t} + w' \frac{\partial \alpha}{\partial z} - \alpha' \frac{\partial u'}{\partial x} - \alpha w' \frac{\partial w'}{\partial z} = 0
\]  
(7.29)

Activity 7.2

Starting from equations 7.21 to 7.24 and applying the assumption made for the gravity waves, derive equations 7.26 to 7.29.

By introducing the following conditions

\[
\alpha' = \alpha_0 e^{-z/\nu}
\]
where

\[ H = R \frac{T}{g} \] is the scale height

\[ \nu = \text{frequency} \]

\[ u' = S \bar{\alpha}^{\frac{1}{2}} e^{i(ux+kz-\nu t)} \]  (7.30)

\[ P' = P \bar{\alpha}^{\frac{1}{2}} e^{i(ux+kz-\nu t)} \]  (7.31)

\[ W' = W \bar{\alpha}^{\frac{1}{2}} e^{i(ux+kz-\nu t)} \]  (7.32)

\[ \alpha' = A \bar{\alpha}^{\frac{1}{2}} e^{i(ux+kz-\nu t)} \]  (7.33)

Substituting equations 7.30 to 7.33 into 7.2 to 29 we obtain

\[ -i \nu \bar{\alpha}^{\frac{1}{2}} S + i u \bar{\alpha}^{\frac{1}{2}} P = 0 \]

\[ -i \nu \bar{\alpha}^{\frac{1}{2}} W + \bar{\alpha} \left( i k \alpha^{\frac{1}{2}} + \frac{1}{2} \alpha^{-\frac{1}{2}} \frac{d \bar{\alpha}}{dz} \right) P - \bar{\alpha}^{\frac{1}{2}} g A = 0 \]

\[ -i \bar{\alpha}^{\frac{1}{2}} \nu P - \bar{\alpha}^{\frac{1}{2}} g W + \phi \bar{\rho} \bar{\alpha}^{\frac{1}{2}} \frac{d \bar{\alpha}}{dz} W - i \nu \bar{\alpha}^{\frac{1}{2}} \phi \bar{\rho} A = 0 \]  (7.34)

\[ \bar{\alpha}^{\frac{3}{2}} \mu i S + i \bar{\alpha}^{\frac{3}{2}} k W + \frac{\bar{\alpha}^{\frac{1}{2}}}{2} \frac{d \bar{\alpha}}{dz} S + i \nu \bar{\alpha}^{\frac{3}{2}} A - \bar{\alpha}^{\frac{1}{2}} \frac{d \bar{\rho}}{dz} W = 0 \]

Dividing through by \( i \bar{\alpha}^{\frac{1}{2}} \) and noting that

\[ \frac{1}{i} = i \frac{1}{l^2} = \frac{i}{-1} = -i \]

we obtain after reorganization

\[ -\nu S + O W + \bar{\alpha} \mu \rho + O A = 0 \]
\[ OS + -\nu W + \bar{\alpha} \left( k - \frac{i}{zH} \right) \rho + gi A = 0 \]

\[ OS + i \left( -g + \frac{\phi RT}{H} \right) W + \bar{\alpha} \nu \rho + \phi \bar{\nu} \nu A = 0 \]

(7.35)

\[ \mu S + \left( k + \frac{i}{zH} \right) W + OP + \nu A = 0 \]

in matrix form

\[
\begin{bmatrix}
-\nu & 0 & \bar{\alpha} \mu & 0 \\
0 & -\nu & 2 \left( k - \frac{i}{2H} \right) & gi \\
0 & i \left( -g + \frac{\phi RT}{H} \right) & \bar{\alpha} \nu & \phi \bar{\nu} \\
\mu & \left( k + \frac{i}{2H} \right) & 0 & \nu \\
\end{bmatrix}
\begin{bmatrix}
S \\
W \\
P \\
A \\
\end{bmatrix}
= 0
\]

(7.36)

The frequency equation is obtained by making the determinant to vanish.

Expanding the determinant and neglecting smaller terms we obtain

\[ \nu^2 \left( \phi RT \right) \left( k^2 + \mu^2 \right) - g^2 \mu^2 \left( \phi - 1 \right) \]

(7.37)

Starting from

\[ \bar{\theta} = \bar{T} \left( \frac{\rho_0}{\bar{\rho}} \right)^{\frac{g'}{\rho'}} \]

It can be shown through logarithm differentiation that

\[ \left( \phi - 1 \right) = \frac{\phi RT}{g} \left( \frac{1}{\bar{\theta}} \frac{\partial \bar{\theta}}{\partial z} \right) \]

(7.35)

Equation 7.37 becomes
\[ \nu^2 \left( \partial R T \right) \left( k^2 + \mu^2 \right) - \varphi R T \mu^2 g \left( \frac{1}{\theta} \frac{\partial \theta}{\partial z} \right) \] 

(7.39)

\[ \nu^2 = \frac{\mu^2}{k^2 + \mu^2} \frac{g}{\theta} \frac{\partial \theta}{\partial z} \]

\[ \nu = \pm \frac{\mu}{\left( k^2 + \mu^2 \right)^{1/2}} \sqrt{\frac{g}{\theta}} \frac{\partial \theta}{\partial z} \] 

(7.40)

7.4.2. Equation for External Gravity Waves

To obtain the equation for external gravity waves we assume the atmosphere to be incompressible and homogenous.

The Gravity Equation

We again consider the motion in \( x - z \) plane

The governing equations are

\[ \frac{du}{dt} = -\alpha \frac{\partial \rho}{\partial x} \] 

(7.41)

\[ \frac{dw}{dt} = -\alpha \frac{\partial \rho}{\partial z} \] 

(7.42)

\[ \nabla \cdot \vec{V} = 0 \] 

(7.43)

Expanding equation 7.41 to 7.43 we get

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\alpha \frac{\partial \rho}{\partial x} \] 

(7.44)

\[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\alpha \frac{\partial \rho}{\partial z} \] 

(7.45)

\[ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \] 

(7.46)

\[ u = U + U^i \]

\[ w = w^i \]
\[ P = \overline{P} + P^i \]

\[ P = \overline{P} = \text{constant} \]

\[ \overline{P} = \text{constant} \]

\[ u^i = \Psi(z) e^{iu(x-c\tau)} \]

\[ w^i = \phi(z) e^{iu(x-c\tau)} \quad (7.47) \]

\[ \frac{P^i}{\rho} = P(z) e^{iu(x-c\tau)} \]

Putting the total field into equation 4 7.44 to 7.46 we obtain

\[ \frac{\partial u'}{\partial t} + u \frac{\partial u'}{\partial x} + \overline{\alpha} \frac{\partial \rho}{\partial x} = 0 \quad (7.48) \]

\[ \frac{\partial w'}{\partial t} + u \frac{\partial w'}{\partial x} + \overline{\alpha} \frac{\partial \rho}{\partial z} = 0 \quad (7.49) \]

\[ \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \quad (7.50) \]

Substituting 7.47 into 7.48 to 7.50 we obtain

\[ -\Psi(z) i \mu e^{iu(x-c\tau)} + U i \mu \Psi e^{iu(x-c\tau)} + \overline{\alpha} \rho i \mu p(z) e^{iu(x-c\tau)} = 0 \]

\[ i \mu c \phi(z) e^{iu(x-c\tau)} + U i \mu \phi e^{iu(x-c\tau)} + \overline{\alpha} \rho \frac{dp}{dz} e^{iu(x-c\tau)} = 0 \]

\[ i \mu \Psi(z) e^{iu(x-c\tau)} + \frac{d\phi}{dz} e^{iu(x-c\tau)} = 0 \quad (7.51) \]

\[ (U - C)\Psi + P(z) = 0 \quad (7.52) \]
\[ i \mu (U - C) \phi + \frac{dp}{dz} = 0 \]  
(7.53)

\[ i \mu \Psi + \frac{d\phi}{dz} = 0 \]  
(7.54)

From 7.52 to 7.54 it can be shown that

\[ \frac{d^2 \phi}{dz^2} - \mu^2 \phi = 0 \]  
(7.55)

**Activity 7.3**

Using equations 7.52, 7.53 and 7.54 derive equation 7.55.

Equation 7.55 is a second order homogenous differential equation with constant coefficient.

**Solution**

\[ \phi = A_1 e^{uz} + A_2 e^{-uz} \]

Using bounding condition

at \( z = 0, w^I = 0 \) so \( \phi(z = 0) = 0 \)

hence \( A_1 = -A_2 = A \)

\[ \phi = A \left( A^{uz} - e^{-uz} \right) \]

\[ = A \sinh (uz) \]  
(7.56)

Boundary condition z.

At \( z = H \) (top of atmosphere)
\[
\frac{d}{dt} \rho = 0 \text{ or } \frac{d}{dt} (\bar{p} + p') \text{ (Dynamic boundary condition)}
\]
(7.57)

Evaluating 7.57 we obtain
\[
-\mu(u - c) - g\rho \phi = 0
\]
(7.58)

Activity 7.4

Use equation 7.57 and 7.47 to obtain 7.58

From equation 7.54
\[
\frac{d\phi}{dz} = -i\mu \Psi
\]
(7.59)

Using equation 7.56 we obtain
\[
\Psi = iA \cos h(\mu, z)
\]
(7.60)

Activity 7.5

1. Use equation 7.56 and 7.59 to reduce equation 7.60.
2. Show that \( p = -iA(u - c) \cosh(\mu z) \)

It can be shown from 7.58 that
\[
\mu \left( u - c \right)^2 = \frac{g}{\mu} \tan h(uz)
\]

\[
C = U \pm \sqrt{\frac{gL}{\mu} \tan h(\frac{2\pi}{L} z)}
\]
where \( \frac{H}{L} \geq 0.5 \)

\[
C \approx U \pm \sqrt{\frac{gL}{2\pi}} \quad \text{(deep waters)}
\]

(7.61)

where \( \frac{H}{L} \leq 0.04 \)

\[
\tanh \left( \frac{2\pi L}{H} H \right) \approx 2\pi \frac{H}{L}
\]

\[
C = U \pm \sqrt{gH} \quad \text{(shallow water)}
\]

(7.62)

### 7.5 Equation for Mixed Gravity Inertia Waves

To obtain the mixed gravity inertia wave we make the following assumptions.

1. The atmosphere is homogenous and incompressible
2. No vertical variation
3. No forcing by ground terrain or diabatic heat source

Here we include the rotation of the earth or coriolis.

The basic equations are the shallow water equations.

\[
\frac{du}{dt} = -g \frac{\partial h}{\partial x} + fv
\]

(7.62)

\[
\frac{dv}{dt} = -g \frac{\partial h}{\partial y} + fu
\]

(7.63)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

(7.64)

we let
\[ u = U + u' \]
\[ v = v' \]
\[ h = H + h' \]
\[ h' = h'(x,t) \]

The third equation can be written in this form

\[
\frac{dh}{dt} + h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0
\]

(7.65)

**Activity 7.5**

Starting from equation 7.64 derive equation 7.65

The linearized perturbation equation

\[
\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} - f v' + g \frac{\partial h'}{\partial x} = 0
\]

(7.66)

\[
\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} - f u + f u' + g \frac{\partial H}{\partial y} = 0
\]

but \( f u + g \frac{\partial H}{\partial y} = 0 \)

\[
\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} + f u' = 0
\]

(7.67)

\[
\frac{\partial h'}{\partial t} + U \frac{\partial h'}{\partial x} + \frac{\partial H}{\partial y} + H \frac{\partial u'}{\partial x} = 0
\]

(7.68)

Let
\[ u' = U_o \ e^{i\omega(x+ct)} \]
\[ v' = V_o \ e^{i\omega(x+ct)} \]
\[ h' = h_o \ \ e^{i\omega(x+ct)} \]
(7.69)

substituting 7.69 in 7.66 to 7.68

\[ C = U \pm \sqrt{gH + \frac{f^2}{\mu^2}} \]
(7.70)

**Activity 7.6**

Starting from equation 7.66 to 7.68 and making use of 7.69 derive equation 7.70

### 7.6 Equation of Inertia Waves

We consider horizontal motion where there is no pressure gradient and friction.

The governing equations are

\[ \frac{du}{dt} = f v \]
(7.71)

\[ \frac{dv}{dt} = -f u \]
(7.72)

We again notice there is no motion in the basic state, thus

\[ \bar{U} = \bar{V} = 0 \]

\[ u = u' \]

\[ v = v' \]
Writing equations 7.71 and 7.72 in perturbation form we have

\[
\frac{du'}{dt} = fv'
\]

(7.73)

\[
\frac{dv'}{dt} = -fu'
\]

(7.74)

\[
u' = \hat{u} e^{ik(x-ct)}
\]

(7.75)

\[
v' = \hat{v} e^{ik(x-ct)}
\]

Substituting gives

\[
i k c \quad \hat{u} + f_0 \hat{v} = 0
\]

(7.76)

\[
f \hat{u} + i k c \hat{v} = 0
\]

From which we find

\[
C = \pm \frac{f}{k}
\]

(7.77)

or

\[
C = \pm \frac{fL}{2\pi}
\]

Activity 7.7

Start from equation 7.76 and derive 7.77

7.7 The Rossby Wave Equation

We assume the flow is horizontal and non-divergent. The vorticity equation thus is
\[
\frac{d}{dt}(\zeta + f) = 0
\]
\[
\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + \beta v = 0
\]
(7.78)

Let

\[
f = f_o + \beta y
\]
\[
\bar{u} = U = \text{constant}
\]
\[
\bar{v} = 0
\]
\[
\zeta = \bar{\zeta} + \zeta^1
\]
\[
v = v'
\]
\[
u = U + u'
\]
\[
\bar{\zeta} = \frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y}
\]
\[
= 0
\]
\[
\therefore \zeta = \zeta^1
\]
\[
= \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}
\]
\[
u' = u'(x)
\]
\[
v' = v'(x)
\]
\[
\therefore \zeta = \frac{\partial v'}{\partial x}
\]

Equation 7.78 after linearization becomes
\[
\frac{\partial}{\partial t} \left( \frac{\partial v'}{\partial x} \right) + U \frac{\partial}{\partial x} \left( \frac{\partial v'}{\partial x} \right) + \beta v' = 0
\]
(7.79)
\[
v' = v e^{ik(x-ct)}
\]
(7.80)

Substituting into 7.79 we obtain
\[
k^2 c v' - U k^2 v' + \beta v' = 0
\]
\[
C = U - \frac{\beta}{k^2}
\]
or
\[
C = U - \frac{\beta L^2}{\pi^2}
\]
(7.81)

**Activity 7.8**

Starting from equation 7.79, work out step by step to obtain equation 7.81.

Other waves such as mixed gravity Rossby waves and Kelvin waves are derived at a masters level.

**REFERENCES**


Prandtl L 1952: Essentials of Fluid Dynamics. Blackie and Son limited


Hare F.K. 1961: The Restless Atmosphere. Hutchinson and Co. LTD
8 Introduction to Lecture Eight

We stated in lecture one that one of the purpose of studying dynamic meteorology is to be able to develop dynamical models. This can be accomplished if we are able to conceptualize the behaviour of the atmosphere. We can either consider the atmosphere to behave as a barotropic system or a baroclinic system.

A barotropic atmosphere is an atmosphere where the density is a function of pressure alone, therefore the density temperature, and pressure surfaces coincides. In a baroclinic atmosphere the density depend on both temperature and pressure and isobaric line do not coincide with isosteric lines.

8.2 Objectives of lecture 8

At the end of this lecture you should be able to:
1. Define a barotropic and a baroclinic atmosphere
2. Derive the potential vorticity equation
3. Discuss the application of potential vorticity
4. Describe the characteristics of a baroclinic atmosphere
5. Describe the link between thermal wind and the selonoid
6. Discuss the concept of available potential energy
7. State the meaning of equivalent barotropic atmosphere
8. Derive the equivalent barotropic level

8.2 Barotropic System

An atmosphere that is always barotropic is referred to as auto-barotropic. The real atmosphere is not always barotropic, but it may be approximated to be nearly barotropic.

A barotropic system has not vertical variation of the horizontal wind. The horizontal equations of motion for a barotropic system is given by:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial h}{\partial x} - fy = 0
\]

(8.1)

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial h}{\partial y} + fu = 0
\]

(8.2)
and the continuity equation take the form

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

(8.3)

The atmosphere near the equator has nearly barotropic.

We approximate \( f \) by

\[ f = \beta y \]

and equation 8.3 is modified in the following manner.

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

\[
\int_{0}^{H+h'} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dz + \int_{0}^{H+h'} \frac{\partial w}{\partial z} dz = 0
\]

If the horizontal divergence is assumed constant in the vertical we have

\[
\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) (H + h') + (w_n + w_0) = 0
\]

for flat ground \( w_0 = 0 \)

and \[ W_n = \frac{dh}{dt} = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \]

The continuity equation becomes

\[
\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} + h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0
\]

or

\[
\frac{\partial h}{\partial t} + \nabla \cdot (h \vec{v}) = 0
\]

The final system of equations are
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \beta vy + g \frac{\partial h}{\partial x} = 0
\]
(8.4)

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \beta uy + g \frac{\partial h}{\partial y} = 0
\]
(8.5)

\[
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hu) + \frac{\partial}{\partial y} (hv) = 0
\]
(8.6)

**Activity 8.1**

- Use equation 8.4 and 8.5 to derive
  - a) Divergence equation
  - b) Vorticity equation

**8.3 Potential Vorticity**

In activity 8.1, you should have showed that the equation of vorticity for a barotropic system is given by

\[
\frac{\partial \eta}{\partial t} + \vec{V} \cdot \nabla \eta + \eta D = 0
\]
(8.7)

where \( \eta = \zeta + f \)

The continuity equation

\[
D = \nabla \cdot \vec{V} = -\frac{\partial w}{\partial z}
\]

\[
\frac{\partial w}{\partial z} = \frac{\partial w}{\partial z}
\]
(8.8)
For barotropic atmosphere $\eta$ is constant with height therefore integrating equation 8.8 from $z_1$ to $Z_z$ we have

$$\int_{z_1}^{z_z} \frac{dn}{dt} \eta \, dz = \int_{z_1}^{z_z} \eta \frac{dw_z}{dz} \, dz$$

(8.9)

$$(Z_z - Z_1) \frac{dn}{dt} \eta = \eta \left(W_{(z_z)} - W_{(z_1)}\right)$$

(8.10)

$$W_{(z_z)} - W_{(z_1)} = \frac{dz_2}{dt} - \frac{dz_1}{dt} = \frac{d}{dt} (Z_z - Z_1)$$

Let $z_z - z_1 = h - H$

$$\frac{d}{dt} (z_1 - z_2) = \frac{dH}{dt} (h - H)$$

Equation 8.10 becomes

$$(h - H) \frac{d\eta}{dt} = \eta \frac{d}{dt} (h - H)$$

(8.11)

From 8.11 we obtain

$$\frac{1}{\eta} \frac{dn\eta}{dt} = \left(\frac{1}{h - H}\right) \frac{d(h - H)}{dt}$$

$$\frac{d}{dt} (\ln \eta) - \frac{d}{dt} (\ln h - H) = 0$$

$$\frac{d}{dt} \ln \left(\frac{\eta}{h - H}\right) = 0$$

or

$$\frac{\eta}{h - H} = \text{constant}$$

(8.12)
\[
\frac{\zeta + f}{h-H}
\] is called potential vorticity.

1. As the current approaches the mountain \((h-H)\) decreases.
2. \(+ f\) must decrease
3. must decrease
4. Negative \(\zeta\) is associated with anticyclonic flow. The flow will turn equatorward.
5. \(f\) decreases.
6. As the flow crosses the mountain, \(H\) decreases and hence \(h-H\) increases.
7. increases until it becomes cyclonic. This means that the flow turns poleward.
8. \(F\) increases.

### 8.4 Baroclinic System

A baroclinic system is one which has strong horizontal temperature gradients, and the presence of horizontal divergence. The baroclinicity is measured by the solenoid intensity.

If the number of \(N\) of \((d, -p)\) – solenoids inside a given rectangular curve \(c\), composed of two isobaric and two vertical curve segments, is to be computed, one starts from the relationship.

\[
N_{s_{d-p}} = -\int_c \alpha \, dp = -R_d \int_c T\delta \left(\ln p\right)
\]

\[(8.13)\]

\[
R_d \int_c T\delta \left(\ln p\right) = -R_d \int_{p_a}^{p_b} T\delta \left(\ln p - R\right)
\]

\[(8.14)\]

where \(\alpha\) is the specific volume of the air and \(R_d\) the gas constant for dry air. The first and the third term in equation 8.13 give no contribution to the integral. Along the vertical segments of the curve \(c\), the integral is identical to the barometric height.

\[
\phi_2 - \phi_1 = R_d T_s m \ln \frac{p_1}{p_2}
\]

\[(8.15)\]

**Activity 8.2**
Starting from the hydrostatic equation derive equation similar to equation 8.15

\( \phi_2, \phi_1 \) in equation 8.15 are the geopotential heights of the lower and upper isobaric surfaces \( \phi_2, \phi_1 \) is the geodynamic thickness of the atmospheric layer of the pressure internal \( p_2, p_1 \). If \( T_{mw} \) and \( T_{mc} \) denote the mean virtue temperatures at the warmer and the colder vertical boundaries of the rectangle, it can be shown that:

\[
|N_{\alpha} - \rho| = R_d (T_{mw} - T_{mc}) \ln \frac{P_1}{P_2} = \phi_2 - \phi_1
\]

This relationship indicates that the number of solenoid between two vertical boundaries of an atmospheric layer with pressure boundaries \( P_1 \) and \( P_2 \) is equal to the dynamic thickness (geopotential height interval) of the layer.

### 8.5 Relationship Between Thermal Wind and Solenoid

The zonal component of geostrophic wind

\[
U_g = -\frac{g}{f} \frac{\partial z}{\partial y}
\]

(8.16)

The thermal wind component of the zonal flow is

\[
\frac{\partial U_g}{\partial p} = -\frac{1}{f} \frac{R}{p} \frac{\partial T}{\partial y}
\]

(8.17)

**Activity 8.3**

Start from equation 8.16 and derive equation 8.17.

The three-dimensional solenoidal vector in the \( z \)-system is

\[
\vec{S} = \nabla \alpha \times (-\nabla \rho)
\]

(8.18)
The component normal to the vertical meridional plane in terms of $\alpha$ and $\rho$ is

$$S_x(\alpha, \rho) = -\frac{\partial \alpha}{\partial y} \left( \frac{\partial \rho}{\partial z} - \frac{\partial \alpha}{\partial z} \frac{\partial \rho}{\partial y} \right)$$

(8.19)

**Activity 8.4**

Derive equation 8.19 from equation 8.18

In equation 8.19, we can replace $\alpha$, with temperature, $T$ using equation of state.

$$S_x(T, \rho) = -\frac{R}{p} \left( \frac{\partial T}{\partial y} \frac{\partial p}{\partial z} - \frac{\partial T}{\partial z} \frac{\partial p}{\partial y} \right)$$

(8.20)

**Activity 8.5**

Derive equation 8.20 starting from equation 8.19

Equation 8.20 can be transformed from the $z$-system to $\rho$-system. The resulting equation is

$$\left( S_x \right)_\rho = \frac{g}{T} \frac{\partial T}{\partial y}$$

(8.21)

Substituting equation 8.21 into equation 8.17 we obtain

$$\frac{\partial U_g}{\partial \rho} = \frac{1}{f} \frac{1}{gp} \left( S_x \right)_\rho$$

(8.22)
The expression in equation 8.22 shows that solenoidal intensity \((S_x)_p\) in the meridional plane is proportional to the thermal wind.

It is clear from the foregoing discussion that the vertical wind shear is also a measure of baroclinicity. Note that the units of solenoidal intensity is time\(^{-2}\) \((S^2)\)

### 8.6 Available potential Energy APE

Available potential energy is the total potential energy \(PE\) of the atmosphere minus potential energy \(PE\) of a hypothetical state in which surfaces of constant pressure and constant density coincide.

In a barotropic atmosphere there is no conversion of \(APE\) to kinetic energy. Only redistribution of energy from one wave number to another take place. The baroclinic atmosphere there is conversion of \(APE\) to \(KE\).

### 8.7 Equivalent Barotropic Atmosphere

An equivalent barotropic is one in which a particular level can be taken as a representative of atmosphere, and which at that level the atmosphere behaves as barotropic.

#### 8.7.1 Determination of the Equivalent Barotropic Level

The governing equation for the barotropic model is

\[
\frac{\partial S}{\partial t} + \vec{V} \cdot \nabla (\zeta + f) = 0
\]

(8.23)

and the requirement for its validity is that the atmosphere should move horizontally without divergence.

Equation 8.23 is customarily applied to the atmosphere at 500 mb. We now show the validity of this application.

The geostrophic horizontal wind is assumed to vary in the vertical as follows:

\[
\vec{V}(x, y, p, t) = A(p) \vec{V}_m(x, y, t)
\]

(8.24)

where \(A(p)\) is an empirical function of pressure describing the vertical variation of the wind speed and where the wind \(\vec{V}_m\) is the vertically averaged wind i.e.
\[ \bar{V}_m = \frac{1}{\rho_0} \int_{\rho_0}^{\rho} \bar{V} \, dp \] (8.25)

**Activity 8.6**

Show using equation 8.25 and 8.24 that \((A(p))_m = 1\)

The implication of \(A(p)\) is that which the wind may change in speed with height, it will always have the same direction, namely the direction of \(\bar{V}_m\). We may also express the same statement in another way. The thermal wind is

\[ \frac{\partial \bar{V}}{\partial \rho} = \frac{dA}{dp} \bar{V}_m \] (8.26)

Now, the thermal wind has thus the same direction as \(\bar{V}_m\), but it is also directed along the isotherms if we assume geostrophy. It follows therefore that the isotherms at any level are parallel to the contours, so that the temperature advection vanishes in an equivalent barotropic atmosphere.

**Activity 8.7**

Show that the temperature advection vanishes in an equivalent barotropic atmosphere.

\[ \zeta = k.\nabla \times \bar{V} = k.\nabla \times (A(p)\bar{V}_m) \]

\[ \zeta = A(p)\zeta_m \] (8.27)

Now consider the vorticity equation
\[
\frac{\partial \zeta}{\partial t} + \mathbf{\tilde{V}} \cdot \nabla (\zeta + f) = f \frac{\partial \omega}{\partial \rho}
\]

(8.28)

Substituting 8.24 and 8.27 into 8.28 we obtain

\[
\frac{\partial}{\partial t} (A(p)\zeta_m) + A(p)\mathbf{\tilde{V}}_m \cdot \nabla A(p)\zeta_m + A(p)\mathbf{\tilde{V}}_m \cdot \nabla f = f \frac{\partial \omega}{\partial \rho}
\]

\[
\frac{\partial}{\partial t} (A(p)\zeta_m) + A^2 \mathbf{\tilde{V}}_m \cdot \nabla \zeta_m + A\mathbf{\tilde{V}}_m \cdot \nabla f = f \frac{\partial \omega}{\partial \rho}
\]

(8.29)

Integrity 8.29 and noting that \( A_m = 1 \) and using the boundary condition \( \omega = 0 \) at \( \rho = 0 \) and \( \omega = 0 \) at \( \rho = \rho_o \).

We have

\[
\frac{\partial \zeta_m}{\partial t} + (A^2) \mathbf{\tilde{V}}_m \cdot \nabla \zeta_m + \mathbf{\tilde{V}}_m \cdot \nabla f = 0
\]

(8.30)

Equation 8.30 is the vorticity equation for the vertical mean flow.

Figure 8.1

The shape of \( A(p) \) is given schematically in figure 8.1 where \( A(p) \) is the full curve with its vertical average of 1.

We define a level \( P_A \) such that

\[
A_s = A(P_A) = (A^2)_m
\]

(8.31)
We then have

\[ \vec{V}_x = A_x \vec{V}_m \]

and \[ \zeta_A = A_A \zeta_m \]  
(8.32)

\[ \zeta_m = \frac{\zeta_A}{A_A}, \vec{V}_m = \frac{\vec{V}_x}{A_x} \]

Substituting into equation 8.30 we have

\[ \frac{1}{A_x} \frac{\partial \zeta_m}{\partial t} \left( \frac{A^2}{A_x^2} \right) \vec{V}_x \nabla \zeta_m + \frac{1}{A_x} \vec{V}_x \cdot \nabla f = 0 \]
(8.33)

\[ (A^2)_m = A_x \]

\[ \frac{1}{A_x} \frac{\partial \zeta_m}{\partial t} + \frac{1}{A_x} \vec{V}_x \nabla \zeta_m + \frac{1}{A_x} \vec{V}_x \cdot \nabla f = 0 \]

multiply through by \( A \), we obtain

\[ \frac{\partial \zeta_m}{\partial t} + \vec{V}_x \cdot \nabla (\zeta_m + f) = 0 \]
(8.34)

This equation shows that the equivalent barotropic model atmosphere behaves as if it were barotropic at the level \( P_A \), which therefore is called the equivalent barotropic level. We notice that this level is located somewhat higher in the atmosphere than the level where the wind is equal to the mean wind.

Thus \( P_m > P_x \)

It is found in practice that \( P_m \) is located at 500 mb, approximately the position of the equivalent barotropic level is in reality a function of location and time, although it is assumed to be located at a constant pressure in each application. The discrepancy may explain some of the errors in the use of equivalent barotropic forecasts.

Equation 8.34 applies only at the equivalent barotropic levels. It is thus possible to have a vertical velocity in an equivalent barotropic atmosphere.
If we substitute for \( \mathfrak{J} \) by \( A\mathfrak{J}_m \) and \( A\vec{V}_m \) for \( \vec{V} \) in equation 8.28 we obtain

\[
A \frac{\partial \zeta_m}{\partial t} + A^2 \vec{V}_m \cdot \nabla \zeta_m + A \vec{V}_m \cdot \nabla f = f \frac{\partial \omega}{\partial \rho}
\]

(8.35)

multiply equation 8.30 by \( A \)

\[
A \frac{\partial \zeta_m}{\partial t} + A(A^2)_m \vec{V}_m \cdot \nabla \zeta_m + A \vec{V}_m \cdot \nabla f = 0
\]

(8.36)

Subtracting 8.36 from 8.35 we obtain

\[
f \frac{\partial \omega}{\partial \rho} = \left[A^2 - A(A^2)_m\right] \vec{V}_m \cdot \nabla \zeta_m
\]

(8.37)

Integrating 8.37 from 0 to \( P \) we have

\[
\omega = \frac{1}{f} \vec{V}_m \cdot \nabla \zeta_m \int_0^P \left[A^2 - A(A^2)_m\right] dp
\]

(8.35)

which can be written as

\[
\omega = c(p) \frac{1}{f} \vec{V}_m \cdot \nabla \zeta_m
\]

where \( c(p) = \int_0^P \left[A^2 - (A^2)_m A\right] dp \)

(8.39)

Activity 8.8

Using equation 8.39
Show that \( C(P_0) = 0 \)
We observe that \( C(P_0) = 0 \) such that the lower boundary condition is satisfied. We note further that \( \omega \) is proportional to the vorticity advection, \( \vec{V}_m . \nabla \zeta_m \)

If we introduce a local co-ordinate system in which the \( s \)-axis is along the steamlines, while \( n \)-axis is along the normal to the streamline, we find

\[
\vec{V}_m . \nabla \zeta_m = V_m \frac{\partial \zeta_m}{\partial s}
\]

(8.40)

and it can be seen that \( \omega \) is positive (sinking motion) when the wind blows from low to high values of the vorticity, while \( \omega \) is negative in the opposite case.

Simple case of \( A(p) \)

\[
A(p) = \begin{cases} 
\frac{2p}{p_T}; & 0 \leq p \leq p_T, \quad p_T = 250m \\
\frac{p_0 - p}{p_0 - p_T}; & p_T \leq p \leq p_0, \quad p_0 = 1000mb 
\end{cases}
\]

(8.41)

**Activity 8.9**

1. Using equation 8.41 show that
   (a) \( A_m = 1 \)
   (b) \( (A^2)_m = 4/3 \)
   (c) \( C(p) \) for \( 0 \leq p \leq p_T \) and \( p_T \leq p \leq p_0 \)

2. Find \( p_m \) and \( p_* \)
   Hint \( p_m = P(A = 1) \) and \( p_* = P(A = (A^2)_m) \)
   At the level of non-divergent \( \frac{\partial \omega}{\partial \rho} = 0 \) hence
   \[
   A^2 - (A^2)_m A = 0
   \]
   or
   \[
   A = (A^2)_m
   \]
In activity 8.9 Q2 did you find

\[ C_{(p)} = \begin{cases} 
-\frac{4}{3} \frac{p^2}{p_T^2} (p_T - p); & 0 \leq p \leq p_T \\
\frac{4}{3} \left( \frac{p_0 - p}{p_0 - p_T} \right)^2 (p - p_T); & p_T \leq p \leq p_0 
\end{cases} \]

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Hare F.K. 1961: The Restless Atmosphere. Hutchinson and Co. LTD
9 DYNAMIC INSTABILITY

9.0 Introduction to Lecture 9
Do you recall from your physics knowledge what we mean when we say that a system is unstable.? If you have forgotten it may be worthy to go back and review. You will notice that a system will be stable or unstable depending on how it responds to a given perturbation. It is therefore the condition of the system that will determine whether it is stable or unstable. In the atmosphere there are different types of instability. The instability that occur by the motions is referred to as dynamic instability. In this lecture we will examine three types of dynamic instabilities; namely, barotropic, baroclinic and inertial instability.

9.1 Objectives of lecture 9

At the end of this lecture you should be able to:

1. State the role played by dynamic instability in generation of the observed weather.
2. State the condition necessary for the occurrence of baroclinic instability
3. Describe the steps followed in deriving the condition for baroclinic instability
4. Explain the meaning of index cycle with reference to baroclinic instability
5. State the condition necessary for the occurrence of barotropic instability.
6. State the difference between a baroclinic and barotropic atmosphere
7. Describe the circumstances under which inertial instability may occur.

9.2 Definition of Instability

If a perturbation is given to an undisturbed atmosphere, if the perturbation grows, then the atmosphere is said to be unstable, if the perturbation disintegrates and dies out then the atmosphere is said to be stable.

The perturbation are assumed to be a sinusoidal wave given by
If $c$ is split into real and imaginary component, then $c = c_r + ic_i$

\[ \sim e^{i\mu(x-ct)} \]

If $c$ is split into real and imaginary component, then $c = c_r + ic_i$

\[ \left( \sim e^{i\mu x} e^{-i\mu(c_r+ic_i)t} \right) \]

\[ \sim e^{i\mu x} e^{-i\mu c_i t} e^{i\mu f} \]

\[ \sim e^{i\mu x} e^{-i\mu(x-c_i,t)} \]

If $c_i > 0$ then the perturbation will grow with time and the atmosphere will become unstable. If $c_i < 0$ then the perturbation will die off with time and the atmosphere is said to be stable. Therefore when we look for instability, we look for conditions that will yield $c_i > 0$.

### 9.3 Baroclinic Instability

Baroclinic instability is the major instability that is responsible for synoptic disturbances, especially in extra-tropical regions.

To derive the condition for this instability we derive the equation for the phase speed. The instability will occur when the phase speed is complex and the imaginary component is positive.

We consider two-level quasi-geostrophic model where the hydrostatic balance is assumed. Further simplification of the primitive equation is made so that only the Rosby waves are permitted. We make use of the vorticity equation and thermodynamic equation.

\[ \frac{\partial \zeta_g}{\partial t} + \nabla_s (\zeta_g + f) = f \frac{\partial \omega}{\partial p} \]

(9.3)

\[ \frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial p} \right) + \nabla_s \nabla \left( \frac{\partial \phi}{\partial p} \right) + \sigma \omega = 0 \]

(9.4)

### Activity 9.1

1. Starting from the equation of the first law of thermodynamic and assuming adiabatic process show that \( \frac{d}{dt} (In\theta) = 0 \)

2. Show that \( \frac{d}{dt} (In\theta) = 0 \) can be expanded to take the form

\[ \frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial p} \right) + \nabla_s \nabla \left( \frac{\partial \phi}{\partial p} \right) + \sigma \omega = 0 \] where \( \sigma = -\alpha \frac{\partial}{\partial p} (In\theta) \)
Equation 9.3 and 9.4 form a closed system. We consider a basic state in which 

\[ \bar{u} = U(p), \quad \bar{v} = \bar{v} = 0. \]

The basic state is therefore a zonal current with vertical wind shear. When we rewrite equation 9.3 and 9.4 in a perturbation form we obtain

\[
\frac{\partial \zeta'}{\partial t} + U \frac{\partial \zeta'}{\partial x} + \beta v' = f \frac{\partial \omega'}{\partial p} 
\]

(9.5)

\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi'}{\partial p} \right) + U \frac{\partial \phi'}{\partial x} + \nu' \frac{\partial \bar{\phi}}{\partial y} + \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial y} + \bar{\sigma} \omega' = 0
\]

(9.6)

Activity 9.2

1. Starting from equation 9.3, derive equation 9.5
2. Starting from equation 9.4 derive equation 9.6

We assume the \( U = U(p) \) is a linear function of \( p \) and that \( \sigma = \sigma(p) = \text{const \ tan} t \)

If we apply equation 9.5 at level 1 and 3 and drop the prime and the subscript \( g \) we have

\[
\frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} + \beta v_1 = f \frac{\partial \omega}{\partial p}
\]

(9.7)

\[
\frac{\partial \zeta}{\partial t} + U_3 \frac{\partial \zeta}{\partial x} + \beta v_3 = f \frac{\partial \omega}{\partial p}
\]

(9.8)

If we let \( \omega_4 = \omega_0 = 0 \) we have the following

\[
\frac{\partial \omega}{\partial p} \bigg|_1 = \frac{\omega_2 - \omega_0}{\Delta p} = \frac{\omega_2}{\Delta p}
\]

(9.9)

\[
\frac{\partial \omega}{\partial p} \bigg|_3 = \frac{\omega_4 - \omega_2}{\Delta p} = \frac{-\omega_2}{\Delta p}
\]

(9.10)

We substitute the right hand of equation 9.7 and 9.8 by 9.9 and 9.10

\[
\frac{1}{2} \frac{\partial \zeta}{\partial t} + \frac{U_1}{2} \frac{\partial \zeta}{\partial x} + \frac{\beta v_1}{2} = \frac{f \omega_2}{2 \Delta p}
\]

9.11

\[
\frac{1}{2} \frac{\partial \zeta}{\partial t} + \frac{U_3}{2} \frac{\partial \zeta}{\partial x} + \frac{\beta v_3}{2} = \frac{-f \omega_2}{2 \Delta p}
\]

9.12

Once we let \( \omega = 0 \) at \( p=0 \), the external gravity waves are eliminated.

Adding 9.11 and 9.12 and reorganizing we obtain

\[
\frac{\partial}{\partial t} \left( \frac{\zeta_1 + \zeta_3}{2} \right) + \left( \frac{U_1 + U_3}{2} \right) \frac{\partial}{\partial x} \left( \frac{\zeta_1 + \zeta_3}{2} \right) + \frac{(U_1 - U_3)}{2} \frac{\partial}{\partial x} \left( \frac{\zeta_1 - \zeta_3}{2} \right) + \frac{\beta (v_1 + v_3)}{2} = 0
\]

9.13
Subtracting 9.12 from 9.11 and reorganizing we obtain

$$\frac{\partial}{\partial t} \left( \frac{\zeta_1 - \zeta_3}{2} \right) + \frac{U_1 + U_3}{2} \frac{\partial}{\partial x} \left( \frac{\zeta_1 - \zeta_3}{2} \right) + \frac{U_1 - U_3}{2} \frac{\partial}{\partial x} \left( \frac{\zeta_1 + \zeta_3}{2} \right) + \beta (v_1 - v_3) = \frac{f \omega}{\Delta p} \quad 9.14$$

If we now introduce the notation

$$x_0 = \frac{1}{2} (x_1 + x_3) \quad \text{and}$$

$$x_T = \frac{1}{2} (x_1 - x_3)$$

We have

$$\frac{\partial \zeta_1}{\partial t} + U_x \frac{\partial \zeta_1}{\partial x} + U_T \frac{\partial \zeta_T}{\partial x} + \beta v_0 = 0 \quad 9.15$$

$$\frac{\partial \zeta_T}{\partial t} + U_x \frac{\partial \zeta_T}{\partial x} + U_T \frac{\partial \zeta_0}{\partial x} + \beta v_T = \frac{f \omega}{\Delta p} \quad 9.16$$

We now turn to equation 9.6 and apply it at level 2 and again drop the prime

$$\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial \phi} \right) + U_x \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial \phi} \right) + v_2 \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial \phi} \right) + \bar{\sigma} \omega_2 = 0 \quad 9.17$$

From the geostrophic equation

$$f U = -\frac{\partial \phi}{\partial y} \quad 9.18$$

$$f \frac{\partial U}{\partial \phi} = -\frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial \phi} \right) \quad 9.19$$

Substituting 9.19 into 9.17 we obtain

$$\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial \phi} \right) + U_x \frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial \phi} \right) - v_2 f \frac{\partial U}{\partial \phi} + \bar{\sigma} \omega_2 = 0$$

On using the finite differencing we obtain

$$\frac{1}{\Delta p} \frac{\partial}{\partial t} \left( \phi - \phi_T \right) + \frac{U_2}{\Delta p} \frac{\partial}{\partial x} \left( \phi - \phi_T \right) - \frac{v_2 f}{\Delta p} (U_3 - U_1) + \bar{\sigma} \omega_2 = 0 \quad 9.20$$

We let

$$U_2 = \frac{U_1 + U_3}{2}, \quad v_2 = \frac{v_1 + v_3}{2} \quad 9.21$$

$$2 \phi_0 = \phi_1 - \phi_3, \quad 2 U_T = U_1 - U_3 \quad \text{we also assume} \quad U_2 \approx U_0 \quad \text{and} \quad v_2 \approx v_0$$

Substituting into equation 9.20 we obtain

$$- \frac{2}{\Delta p} \frac{\partial \phi_T}{\partial t} - \frac{2U_x}{\Delta p} \frac{\partial \phi_T}{\partial x} + \frac{2U_T f v_T}{\Delta p} + \bar{\sigma} \omega_2 = 0 \quad 9.23$$
Or
\[ \omega = \frac{2}{\sigma \Delta p} \left[ \frac{\partial \phi_r}{\partial t} + U_\tau \frac{\partial \phi_r}{\partial x} - U_\tau f v_r \right] \]  \hspace{1cm} (9.24)

Now
\[ \zeta_s = \frac{1}{f} \frac{\partial^2 \phi_s}{\partial x^2} ; \quad \zeta_T = \frac{1}{f} \frac{\partial^2 \phi_T}{\partial x^2} \]  \hspace{1cm} (9.25)

We note further that
\[ v_s = \frac{1}{f} \frac{\partial \phi_s}{\partial x} ; \quad v_T = \frac{1}{f} \frac{\partial \phi_T}{\partial x} \]  \hspace{1cm} (9.26)

Substituting equations 9.24, 9.25, 9.26 into 9.15 and 9.16 we obtain
\[ \frac{1}{f} \frac{\partial}{\partial t} \left( \frac{\partial^2 \phi_x}{\partial x^2} \right) + U_s \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_x}{\partial x^2} \right) + U_T \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_x}{\partial x^2} \right) + \beta \frac{\partial \phi_x}{\partial x} = 0 \] ....... (27)
\[ \frac{1}{f} \frac{\partial}{\partial t} \left( \frac{\partial^2 \phi_T}{\partial x^2} \right) + U_s \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_T}{\partial x^2} \right) + U_T \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_T}{\partial x^2} \right) + \beta \frac{\partial \phi_T}{\partial x} = \frac{2f}{\sigma (\Delta p)^2} \left[ \frac{\partial \phi_T}{\partial t} + U_s \frac{\partial \phi_T}{\partial x} - U_T \frac{\partial \phi_T}{\partial x} \right] \] ....... (28)

Let \( q_x^2 = \frac{2f^2}{\sigma (\Delta p)^2} \) ....... (29)

Reorganizing equation 9.27 and 9.28 we obtain
\[ \frac{\partial}{\partial t} \left( \frac{\partial^2 \phi_x}{\partial x^2} \right) + U_s \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_x}{\partial x^2} \right) + U_T \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_x}{\partial x^2} \right) + \beta \frac{\partial \phi_x}{\partial x} = 0 \] ....... (30)
\[ \frac{\partial}{\partial t} \left( \frac{\partial^2 \phi_T}{\partial x^2} \right) + U_s \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_T}{\partial x^2} \right) + U_T \frac{\partial}{\partial x} \left( \frac{\partial^2 \phi_T}{\partial x^2} \right) + \beta \frac{\partial \phi_T}{\partial x} = \frac{2f}{\sigma (\Delta p)^2} \left[ \frac{\partial \phi_T}{\partial t} + U_s \frac{\partial \phi_T}{\partial x} - U_T \frac{\partial \phi_T}{\partial x} \right] \] ....... (31)

We let the perturbation have the form
\[ z = \hat{z} e^{i(k(x - ct))} \] ....... (32)

When expression 9.32 is introduced into equation 9.30 and 9.31 we obtain
\[ (c - U_\tau + c_R) \hat{\phi}_x - U_T \hat{\phi}_T = 0 \] ....... (9.33)
\[ \left( \frac{q_x^2}{k^2} - 1 \right) U_\tau \hat{\phi}_x + \left( c - U_\tau + c_R + \frac{q_x^2}{k^2} \right) \left( c - U_\tau \right) \hat{\phi}_T = 0 \] ....... (9.34)

Where \( c_R = \frac{\beta}{k^2} \)

We let \( x = c - U_\tau \) and solve for \( x \) in equation 9.33 and 9.34. \( x \) is the speed of the wave relative to the vertically averaged wind. The solution is obtained by equating the determinant of the matrix formed by the set of linear equations in 9.33 and 9.34 to zero. Thus
\[ (1 + \frac{q_x^2}{k^2}) x^2 + (2 + \frac{q_x^2}{k^2}) c_R x + (c_R^2 - (1 - \frac{q_x^2}{k^2}) U_T^2 = 0 \] ....... (9.35)
Solving for \( x \) we get
\[
x = -\frac{(2 + \frac{q^2}{k^2^2})}{2(1 + \frac{q^2}{k^2^2})} c_R \pm \sqrt{\frac{q^2}{k^2^2} c_R^2 + 4(1 - \frac{q^2}{k^2^2}) U_T^2}{2(1 + \frac{q^2}{k^2^2})} \quad \text{......... (9.36)}
\]
Which gives the wave speed of the waves which are possible in quasi-geostrophic two level model.

For instability we require that \( x \) be a complex number, thus
\[
\frac{q^2}{k^2^2} c_R^2 + 4(1 - \frac{q^2}{k^2^2}) U_T^2 < 0 \quad \text{......................... (9.37)}
\]
This condition can only be satisfied if
\[ k < q \quad \text{or} \quad \frac{2 \pi}{k} < q \quad \text{thus} \quad \lambda > \frac{2 \pi}{q} \quad \text{......................... (9.38)}
\]
Let \( \frac{2 \pi}{q} = \lambda_c \). It can be seen that sufficiently short waves are always stable
\[
\lambda_c^2 = \frac{4 \pi^2 \sigma_T \Delta \rho}{2 f^2} \quad \text{......................... (9.39)}
\]

Activity 9.1

Compute the value of \( \lambda_c \) if \( \sigma = 2 \times 10^{-6} \text{mKs} \) units, \( \Delta \rho = 500 \text{mb} \) at the following latitudes;
10°N, 20°N, 30°N, 40°N, 50°N, and 60°N
What do you notice about the value of \( \lambda_c \) as you move equatorward.

From equation 9.37 the condition for instability is
\[
U_T > \frac{\frac{1}{2} \frac{q^2}{k^2^2} c_R}{\sqrt{\left(\frac{q^2}{k^2^2} - 1\right)}} = U_{T,c} \quad \text{......................... (9.40)}
\]
Where \( U_{T,c} \) is the critical thermal wind.

If we consider mid-latitude we find that instability will occur for \( U_T > 4 \text{ms}^{-1} \). Since for most cases in the mid-latitude \( U_T \approx 7.3 \text{ms}^{-1} \), we see that the westerlies in the middle latitude are always baroclinically unstable.

The vertical wind shear are proportional to horizontal temperature gradient. Since the thermal wind is proportional to horizontal temperature gradient, the greater the thermal wind, the more likely will the unstable situation occur.

The horizontal temperature gradient are greater in the mid-latitude in winter and hence baroclinic instabilities are more intense in winter compared to summer.

### 9.4 Index Cycle

We start with undisturbed situation in summer, as the winter approaches, there is strong horizontal temperature gradients developing due to horizontal temperature gradients developing due to horizontal differential heating. The thermal wind relationship suggest that the vertical wind shear must increase. The baroclinic instability is intensified, this creates extra-tropical cyclones which creates substantial meridional component. The
meridional components redistribute the energy hence reducing the horizontal temperatures. This decreases the thermal wind and hence the intensity of instability; and the meridional flow. The reduced meridional flow allows the differential heating to take place and the cycle begins again. The baroclinic instability is responsible for the extratropical cycle and frontal systems.

9.5 Barotropic Instability
It is of interest to demonstrate whether disturbances can amplify or dissipate (decay) in a barotropic atmosphere. Since the pressure and temperature surfaces coincide under barotropy, all pressure surfaces will be parallel and the wind velocity will not vary in the vertical.

Starting from
\[ \frac{\partial \zeta}{\partial t} + \vec{V} \nabla (\zeta + f) = \vec{f} \frac{\partial \phi}{\partial \rho} \]
\[ \text{....................... 9.41} \]

The vertical integration of 9.41 yields
\[ \frac{\partial \zeta}{\partial t} + \vec{V} \nabla (\zeta + f) = 0 \]
\[ \text{....................... 9.42} \]

For simplicity, the vertical boundary condition are taken as \( \omega \) at \( p = p_o \) and \( p = 0 \)

If we assume \( U = U(y) \), and introduce perturbation in equation 9.42, we obtain after linearization
\[ \frac{\partial \zeta'}{\partial t} + U \frac{\partial \zeta'}{\partial x} + v' \left( \frac{\partial \zeta}{\partial y} + \beta \right) = 0 \]
\[ \text{....................... 9.43} \]

Now we replace \( \zeta' \) by \( \nabla^2 \psi \) and \( \zeta \) by \(- \frac{\partial U}{\partial y}\) we obtain
\[ \frac{\partial (\nabla^2 \psi)}{\partial t} + U \frac{\partial \nabla^2 \psi}{\partial x} + \left( \beta - \frac{\partial^2 U}{\partial y^2} \right) \frac{\partial \psi}{\partial x} = 0 \]
\[ \text{....................... 9.44} \]

Let
\[ \psi = \Psi'(y) e^{i\mu(x-ct)} \]
\[ \text{....................... 9.45} \]

Substituting equation 9.45 into 9.44 and integrating it in the channel from \(-d\) to \(d\), it can be shown that for \( c_i \) not to be equal to zero \( \frac{d^2 U}{dy^2} - \beta \) must change signs at least once in the region \(-d < y < d\). Thus a necessary condition for barotropic instability is that at some value(s) of \( y \), say \( y_k \),
\[ \left( \frac{d^2 U}{dy^2} - \beta \right)_{y_k} = 0, \text{ } -d < y_k < d \]
\[ \text{....................... 9.46} \]

This theorem was originally derived by Lord Rayleigh for a non rotating system. It was later extended by Kuo (1951) for meteorological applications to a rotating earth by the addition of the \( \beta \) term.

We may rewrite equation
\[
\frac{d}{dy} \left( -\frac{dU}{dy} + f \right) = 0 \quad \text{or} \quad \frac{d\zeta}{dy} = 0 \quad \text{..................} \quad 9.47
\]

Which states that for barotropic instability to occur the absolute vorticity must be a maximum or minimum at some points in the basic current.

The phase speed is
\[
c = c_r + ic_i
\]

The phase velocity of neutral waves can never exceed the maximum wind speed, but may be less than the minimum wind speed
\[
c = U - \frac{\beta \chi^2}{4\pi^2}
\]

For amplified waves to exist, the absolute vorticity must have a maximum or minimum within the belt and if no such point exist, all waves with a phase velocity greater than the minimum wind speed will be damped.

The amplifying waves are generally slow but with large wavelength, while non-amplifying waves move fast and have small wavelength.

**9.5 Inertial Instability**

The absolute vorticity in the northern hemisphere is positive and negative in the southern hemisphere. When this condition is violated, then inertial instability will occur.
\[
\zeta + f > 0 \quad \text{northern hemisphere}
\]
\[
\zeta + f < 0 \quad \text{southern hemisphere}
\]

Usually the absolute value of the coriolis parameter is greater than the absolute value of the relative vorticity.

In the upper levels, the motion is largely zonal and geostrophic. If we consider the case of a westerly jet stream.

\[
\eta_g = f - \frac{\partial u_g}{\partial y}
\]

The condition of unstable stability is likely to occur on the equatorward side. It can also be shown that for the southern hemisphere the unstable stability occur on the equatorward side of the westerly jet.
ACTIVITY 9.2

1. Which side of the easterly jet stream will unstable stability occur

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LECTURE 10
NUMERICAL WEATHER PREDICTION (NWP)

10.0 Introduction

Numerical weather prediction (NWP) is the prediction of weather from the present state to future state using the equations of motion in finite difference form. In this lecture you will learn about the basis for numerical weather prediction, numerical integration schemes, Initialization, the boundary conditions, problems of NWP in the tropics and examples of NWP models.

To predict the future state of the atmosphere from the present state we require the following:

1. A system of equations governing the state of the atmosphere.
2. A method of expressing the governing equations in finite difference form
3. A method of integrating the system of equations governing the atmosphere in time in order to obtain the future values of these variables.
4. Choice of the vertical coordinate and map-projection.
5. The initial data of the atmospheric variables such as wind, humidity, temperature, pressure etc.
6. The boundary conditions; bottom, top and lateral
7. A method of including the external factors and sub-scale processes.
8. High speed computer and software for processing the output products
9. Model validation

10.1 Objectives of Lecture 10

At the end of this lecture you should be able to:

1. State what numerical weather prediction is
2. Discuss the merits and demerits of the various integration schemes
3. Integrate a differential equation numerically
4. Discuss the importance of initialization in NWP
5. Describe the steps followed in static and dynamic initialization.
6. State the role of boundary conditions in NWP
7. Discuss the problems of NWP in the tropics
8. Describe the formulation of an equivalent barotropic model
9. State the assumption main features of a quasi-geostrophic baroclinic models

10.2 The Governing Equations

The principal equations which are relevant to motions in the atmosphere are the following:

(a) Newton’s second law of motion
(b) The first law of thermodynamics
(c) The law of conservation of mass or continuity equation
(d) The equation of state
(e) The conservation equation for water substance

The equation representing Newton’s second law of motion is given by
\[ \frac{d\vec{V}}{dt} = -\alpha \nabla p - 2\tilde{\Omega} \times \vec{V} + \tilde{g} + \vec{F} = 0 \] ........................... 10.1
Where \( \alpha \) is the specific volume and \( \tilde{g} \) is the sum of gravitation and centrifugal force \( \tilde{\Omega} \times (\tilde{\Omega} \times \vec{r}) \).

The second term on the right is coriolis force and the last term the frictional force. The equation for the first law of thermodynamic is
\[ \frac{dh}{dt} = c_p \frac{dT}{dt} - \alpha \frac{dp}{dt} \] ........................... 10.2
Where \( \frac{dh}{dt} \) is the rate of heat addition

The continuity equation (law of conservation of mass) is given by
\[ \frac{\partial \rho}{\partial t} + \nabla . \rho \vec{V} = 0 \] ........................... 10.3

Activity 10.1

1. Show that equation 10.3 can be written as \( \frac{1}{\alpha} \frac{d\alpha}{dt} = \nabla \vec{V} \)
2. Split equation 10.1 into three components, thus i, j and k components

The equation of state
\[ p\alpha = RT \] .......................... 10.4
Where \( R \) is the gas constant
\[ R = R_d (1 + 0.61q) \] .......................... 10.5
\( R_d \) is the gas constant for dry air and \( q \) is the specific humidity.

The equation for the conservation of water vapour
\[ \frac{dq}{dt} = (E - P) + D \] .......................... 10.6
\( D \) is the source term due to horizontal and vertical mixing

**10.2.1 The Filter Problem**
The form equations 10.1 to 10.6 takes depends on the system being modeled. In Lecture 8 we saw that the equations of motion supports all forms of motions, some of which do not have meteorological significant. Therefore before integration is done, approximation is made to the above equations so that only meteorological waves are allowed.
The external gravity waves are excluded by adopting a modified lower boundary condition that \( \omega(p_o) = 0 \).

The filter problem requires a considerable meteorological insight to design a set of equations which are restricted in such a way that all unimportant phenomena are removed, while the essential features are left with slight modification. One method of filtering is the so called scale analysis. Scale analysis results in omitting terms of small order of magnitude. However, the resulting equations do not capture some processes that are important in the observed weather such as the meso-scale systems such as the land-sea breeze, or the development of clouds. The small scale systems such as the development of clouds are however included through the process referred to as parameterization. In parameterization the processes are computed in a routine and the output included in the model. In other words these processes are modeled separately from the main model. An example of parameterization is the cumulus parameterization.

10.3 Finite Difference

In order to perform numerical integration, the differential equations must be written infinite difference form. In lecture one you learnt about the various differencing schemes. If you have forgotten, you may need to review them. The three basic differencing schemes are, namely; backward, forward and centred.

Activity 10.2

1. Describe how the backward, forward and centred differencing scheme are derived.

2. Find the following at \( x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2} \):
   
   a) \( \frac{d(\cos x)}{dx} \) \quad b) \( \frac{d(\sin x)}{dx} \) Using
   
   i) Forward \quad ii) backward and iii) centred differencing for \( \Delta x = \frac{\pi}{50}, \frac{\pi}{100}, \frac{\pi}{500}, \frac{\pi}{1000} \)
   
   What do you notice as \( \Delta x \to 0 \)

3. Write the following differential equations in finite difference form using; forward, backward and centred:
   
   a) \( \frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = 0 \)
   
   b) \( \frac{\partial G}{\partial t} + \kappa G = 0 \)

10.3.1 Time Level

What difficulty did you encounter in solving problem 3 in activity 10.2. You may have noticed that when you deal with the time differencing, the question arise as to what time level do you write the other terms.

Consider the equation

\[ \frac{\partial f}{\partial t} = F \] \hspace{1cm} \text{............................... (10.7)}

Where \( F \) is dependent on time.
Forward differencing gives
\[
\frac{f(t + \Delta t) - f(t)}{\Delta t} = F(t) \quad \text{………………………… (10.8)}
\]

The backward differencing gives
\[
\frac{f(t + \Delta t) - f(t)}{\Delta t} = F(t + \Delta t) \quad \text{………………… (10.9)}
\]

The centred differencing gives
\[
\frac{f(t + \Delta t) - f(t - \Delta t)}{2\Delta t} = F(t) \quad \text{………………………… (10.10)}
\]

There is a fourth differencing scheme, referred to as trapezoidal which gives
\[
\frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{1}{2}(F(t + \Delta t) + F(t)) \quad \text{………………… (10.11)}
\]

You will notice that when you are dealing with forward, backward and trapezoidal differencing schemes, the variables are evaluated at two time levels, namely \( t + \Delta t \) and \( t \), while the centred differencing schemes (which is sometimes referred to as leap frog), the variables are evaluated at three time levels namely, \( t + \Delta t \), \( t \), and \( t - \Delta t \). As it was noted in lecture 1, the centred differencing is preferred for space differencing. The reason for this choice is that it has the least truncation error.

Activity 10.3

Find the truncation error in using each of the following schemes
i) forward
ii) backward
iii) centred
iv) trapezoidal

to evaluate \( \frac{df}{dx} \) at the point \( x = 2 \) taking \( \Delta x = 0.2 \), \( \Delta x = 0.05 \), and \( \Delta x = 0.02 \)

for the following
a) \( f(x) = 4x - 8 \)
b) \( f(x) = 8x - 2x^2 \)

In the next section we are going to see the reason for choosing a particular scheme for integration.

**10.4 Integration Scheme**

Integration is the reverse of differentiation. However, since the differential of a constant is always zero, in the analytical problems, the initial conditions are used to
determine the constants. Although the numerical integration makes use of the initial values, there are other problems that need to be address, these include the convergence of the numerical solution to the analytical (exact) value.

10.4.1 Convergence of the Numerical Solution

Consider the equation
\[ \frac{dQ}{dt} = -\kappa Q \] .......................... 10.12

The analytical solution is
\[ Q = Q_0 e^{-\kappa t} \] .......................... 10.13

Activity 10.4

Find the analytical solution of
a) \[ \frac{dG}{dt} = \lambda G \]
b) \[ \frac{dx}{dt} = -\omega^2 x \]

If we write equation 10.12 in the finite form using the backward scheme we get
\[ \frac{Q(t + \Delta t) - Q(t)}{\Delta t} = -\kappa Q(t + \Delta t) \] .......................... 10.14

Reorganizing equation 10.4 we obtain
\[ Q_{(t+\Delta t)} \left(1 + \kappa \Delta t\right) = Q(t) \] .......................... 10.15
Or
\[ Q_{(t+\Delta t)} = \frac{1}{\left(1 + \kappa \Delta t\right)} Q(t) \]

If we are starting from \( t_o \) and integrating to time \( t \), and if we chose \( n \) steps, then
\[ \Delta t = \frac{t - t_o}{n} \]
The first step is
\[ Q_{(t_o + \Delta t)} = \frac{1}{\left(1 + \kappa \Delta t\right)} Q_{(t_o)} \]
The second step
\[ Q_{(t_o + 2\Delta t)} = \frac{1}{\left(1 + \kappa \Delta t\right)^2} Q_{(t_o)} \]
Third step
\[ Q_{(t_o + 3\Delta t)} = \frac{1}{\left(1 + \kappa \Delta t\right)^3} Q_{(t_o)} \]
The \( n^{th} \) step
\[
Q_{(t_{n} + n\Delta t)} = \frac{1}{(1 + \kappa \Delta t)^n} Q_{(t_{n})}
\]

Expanding the coefficient on the right hand side we have
\[
(1 + \kappa \Delta t)^n = 1 + (-n)(-1)\frac{(\kappa \Delta t)^2}{2} + (-n)(-1)(-2)\frac{(\kappa \Delta t)^3}{6} + ...
\]
\[
= 1 + \kappa n \Delta t + \left[ n^2\frac{(\kappa \Delta t)^2}{2} + n\frac{(\kappa \Delta t)^3}{6} + \right] - n(n+1)(n+2)\frac{(\kappa \Delta t)^3}{6} + ...
\]
\[
\text{……… 10.16}
\]

The analytical solution is
\[
Q = Q_{o} e^{-\kappa t}
\]
If we let \( t = n\Delta t \), then we have
\[
Q = Q_{o} \left[ 1 - \kappa n \Delta t + \frac{(\kappa n \Delta t)^2}{2} - \frac{(\kappa n \Delta t)^3}{6} + \right]
\]
\[
\text{…………………………. 10.17}
\]

Comparing the numerical solution and the analytical solution we find that the first and second terms are the same. Comparing the third term we see that if \( \Delta t \) is very small it tends to \( \frac{n^2\kappa^2\Delta t^2}{2} \) which is the same as the analytical solution.

Activity 10.5

Use the forward and trapezoidal scheme to find the solution of \( \frac{dQ}{dt} = -\kappa Q \) and compare the numerical solution in each case to the analytical solution.

We now consider the leap frog integration scheme
\[
\frac{Q_{(t+\Delta t)} - Q_{(t-\Delta t)}}{2\Delta t} = -\kappa Q_{(t)}
\]
\[
\text{……………………… 10.18}
\]

Or
\[
Q_{(t+\Delta t)} + 2\Delta t Q_{(t)} - Q_{(t-\Delta t)} = 0
\]
\[
\text{………………………… 10.19}
\]

Assume \( Q(t) = ce^{rt} \)

The equation 10.19 becomes
\[
\left( e^{r\Delta t} + 2\kappa \Delta t - e^{-r\Delta t} \right) e^{r\Delta t} = 0
\]
\[
\left( e^{r\Delta t} - e^{-r\Delta t} \right) e^{r\Delta t} = -\kappa \Delta t
\]
\[
\text{sinh}(r\Delta t) = -r\Delta t
\]

We see that if we let \( r \Delta t = \pm \alpha \) then the solution is to 10.20
Hence \( Q(t) = c_1 e^{ct} + c_2 e^{-ct} \)  ............10.21

We know that the analytical solution resembles the second term of 10.21. The first term therefore represent the computational mode. We see that although the leap frog is best for differentiation when use in integration it generates a computational mode.

### 10.4.2 Computational Stability

The replacement of the analytical equation by finite difference equation gives rise to one order higher, thus it gives rise to two solutions, one corresponding to the analytical mode and the other is the computational mode. If the computational mode is unstable it will grow and obscure the result.

Consider the linearized vorticity equation

\[
\frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} = 0 \tag{10.22}
\]

The analytical solution of 10.22 can be obtained by using the method referred to as separation of variables

Let \( \zeta = G(t)H(x) \)  ................. 10.23

Then

\[
\frac{\partial \zeta}{\partial t} = H(x) \frac{\partial G}{\partial t} = H(x) \frac{dG}{dt} \tag{10.24}
\]

\[
\frac{\partial \zeta}{\partial x} = G(t) \frac{\partial H}{\partial x} = G(t) \frac{dH}{dx} \tag{10.25}
\]

Substituting 10.24 and 10.25 into 10.22 we have

\[
H(x) \frac{dG}{dt} + UG(t) \frac{dH}{dx} = 0 \tag{10.26}
\]

Dividing equation 10.26 by \( G(t)H(x) \) we obtain

\[
\frac{1}{G(t)} \frac{dG}{dt} + U \frac{1}{H(x)} \frac{dH}{dx} = 0 \tag{10.27}
\]

or

\[
\frac{1}{G(t)} \frac{dG}{dt} = -U \frac{1}{H(x)} \frac{dH}{dx} = -k \tag{10.28}
\]

Where \( k \) is the separation constant

\[
\frac{dG}{dt} + kG(t) = 0 \Rightarrow G(t) = A' e^{-kt} \tag{10.29}
\]

\[
\frac{dH}{dx} - k \frac{H(x)}{U} = 0 \Rightarrow H(x) = B e^{kx} \tag{10.30}
\]

Hence \( \zeta = A e^{k(x-ct)} \)  ................. 10.31

Where \( A = A'B \)

What is the condition for computational stability for the numerical scheme?
We will consider the forward, backward and trapezoidal before considering the leap frog.

\[ \frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} = 0 \]  

10.33

Assume that

\[ \zeta = B(t) e^{i \alpha x} \]  

10.34

Differentiating with respect to time, \( t \) we get

\[ \frac{\partial \zeta}{\partial t} = \frac{dB}{dt} e^{i \alpha x} \]  

10.35

Differentiating with respect to space, \( x \) we get

\[ \frac{\partial \zeta}{\partial x} = B(t)i \mu e^{i \alpha x} \]  

10.36

Substituting 10.35 and 10.36 into 10.33 we obtain

\[ 0 = \frac{dB}{dt} e^{i \alpha x} + U i \mu B e^{i \alpha x} \]  

10.37

Or

\[ \frac{dB}{dt} = -i \omega B \]  

10.38

Where \( \omega = U \mu \)

Letting \( n \) represent the time level and writing equation 10.38 in finite difference form we obtain

\[ \frac{B_{n+1} - B_n}{\Delta t} = i \omega B_n \]  

10.39

The question is whether we should take \( B \) on the right hand side at \( n+1 \) or \( n \). For now let us replace it with \( aB_{n+1} + bB_n \)

\[ \frac{B_{n+1} - B_n}{\Delta t} = i \omega (aB_{n+1} + bB_n) \]  

10.40

Solving equation 10.40 for \( B_{n+1} \) we obtain

\[ B_{n+1} = \left( \frac{1 - ib \omega \Delta t}{1 + ib \omega \Delta t} \right) B_n \]  

10.41

For computational stability

\[ \frac{|1 - ib \omega \Delta t|}{|1 + ib \omega \Delta t|} \leq 1 \]  

10.42

Or \( b \leq a \)  

10.43

Activity 10.5

1. Starting from equation 10.42, show that for computational stability \( b \leq a \)

Case 1

\( a = 0, \ b = 1 \) we have the forward differencing scheme and equation 10.41 becomes

\[ B_{n+1} = (1 - i \omega \Delta t) B_n \]  

10.44

Let \( \lambda = 1 - i \omega \Delta t \)
We see that $|\lambda| > 1$
Hence the solution is amplified with time. Therefore the forward scheme is always unstable. If there is an error in the initial data, it will grow with each integration time step.

Case 2
When $a = 1$, and $b = 0$ we have the backward differencing scheme and equation 10.41 becomes
$$B_{n+1} = \frac{1}{(1 + i\omega \Delta t)} B_n$$

If we let
$$\frac{1}{(1 + i\omega \Delta t)} = \lambda$$
We see that $|\lambda| < 1$, hence the solution is damped with time. Therefore the backward differencing scheme is always stable. If there is error in the initial data, it is suppressed with time.

Case 3
When $a = b = \frac{1}{2}$, we have the trapezoidal differencing scheme, and equation 10.41 becomes
$$B_{n+1} = \frac{(1 - \frac{1}{2}i\omega \Delta t)}{(1 + \frac{1}{2}i\omega \Delta t)} B_n$$

Or
$$B_{n+1} = \frac{(2 - i\omega \Delta t)}{(2 + i\omega \Delta t)} B_n$$

We see that in this case $|\lambda| = 1$ Therefore there is no change in amplitude of the solution and hence the error in the initial data neither amplifies nor decay with each time step. Now we turn to the leap frog scheme.

Assume that solution is
$$\zeta = B' e^{i\mu x}$$

If we let $t = n\Delta t$ and $x = m\Delta x$, then equation 10.49 becomes
$$\zeta_{m,n} = B'^{n\Delta t} e^{i\mu n\Delta x}$$

Turning to the linearized vorticity equation
$$\frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} = 0$$
In finite difference form we have

\[ \frac{\zeta_{m,n+1} - \zeta_{m,n}}{2\Delta t} + \frac{U}{2\Delta x} (\zeta_{m+1,n} - \zeta_{m-1,n}) = 0 \] .......................... 10.52

Substituting 10.50 into 10.52 and multiplying through by \(2\Delta t\)

We obtain

\[ B^{(n+1)\Delta t} e^{ij\mu\Delta x} - B^{(n-1)\Delta t} e^{-ij\mu\Delta x} + \frac{U\Delta t}{\Delta x} (B^{n\Delta t} e^{ij\mu(n+1)\Delta x} - B^{n\Delta t} e^{-ij\mu(n-1)\Delta x}) = 0 \] .......................... 10.53

Dividing through by \(B^{(n-1)\Delta t} e^{ij\mu\Delta x}\) we obtain

\[ B^{2\Delta t} + \frac{U\Delta t}{\Delta x} (e^{ij\mu\Delta x} - e^{-ij\mu\Delta x}) B^{\Delta t} - 1 = 0 \] .......................... 10.54

\[ e^{ij\mu\Delta x} = \cos \mu \Delta x + i \sin \mu \Delta x \] .......................... 10.55

\[ e^{-ij\mu\Delta x} = \cos \mu \Delta x - i \sin \mu \Delta x \] .......................... 10.56

\[ e^{ij\mu\Delta x} - e^{-ij\mu\Delta x} = 2i \sin \mu \Delta x \] .......................... 10.57

Substituting equation 10.57 into 10.54 we obtain

\[ B^{2\Delta t} + 2 \frac{U\Delta t}{\Delta x} i \sin \mu \Delta x B^{\Delta t} - 1 = 0 \] .......................... 10.58

Let \( \frac{U\Delta t}{\Delta x} \sin \mu \Delta x = \sigma \)

Then

\[ B^{2\Delta t} + 2i \sigma B^{\Delta t} - 1 = 0 \] .......................... 10.59

The solution for \(B^{\Delta t}\) is

\[ B^{\Delta t} = \frac{-2i \sigma \pm \sqrt{(2i \sigma)^2 + 4}}{2} \] .......................... 10.60

Or \(B^{\Delta t} = -i \sigma \pm \sqrt{1 - \sigma^2}\)

For stability \(\sigma^2 \leq 1\)

Thus \(\frac{U\Delta t}{\Delta x} \leq 1\) .......................... 10.61

The condition in equation 10.61 is called Courant Fredrick and Lewy CFL condition for computational stability

**10.4 Choice of Vertical Coordinate**

In constructing a numerical model of the atmosphere, one of the first decision to make is the choice of the vertical coordinate. Several systems have been used besides the \(Z\) system such as the \(p\) (pressure) system, the \(\theta\) system (potential temperature), and the \(\sigma\) system where \(\sigma = p/p_0\), and \(p_0\) is the surface pressure and \(p\) the pressure at any level. The most common system presently used in numerical modeling is the \(\sigma\) system. With this system the problem of having a vertical system that intersects the mountain is eliminated,
because at $\sigma=1$, refers to the surface irrespective of its elevation. Furthermore, with this system it is easier to incorporate the vertical exchange processes in the planetary boundary layer.

Other possible vertical coordinate are

$$\sigma = \frac{p_t - p}{p_s - p_t} \quad \text{where the top of the atmosphere is fixed by } p_t$$

$$\sigma = \frac{Z - Z_s}{Z_t - Z_s} \quad \text{where } Z_t \text{ is the height of the top level and } Z_s \text{ is the height of the surface.}$$

### Activity 10.6

1. If the model uses the $\sigma=p/p_s$ coordinate, and the surface level is 820mb, compute the levels for which $\sigma=0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2$ and 0.1
2. Why do you think it is necessary to have levels that are close together near the surface.
3. What is the advantage of using the pressure levels in analysis of NWP outputs instead of $z$

### 10.5 Initialization

One of the steps in initialization is to have the observed data which are usually at the stations, unevenly spaced interpolated unto grid points. This in itself may generate some errors into the initial data. The other sources of error are, truncation and observational errors.

The convectional analyses of wind and pressure data contain large imbalances between the pressure and coriolis forces which can lead to large-amplitude inertial-gravity waves if the latter are permitted in the prediction equations.

Most numerical weather prediction models assume hydrostatic balance. This is not valid for turdadoes and tropical cyclones. The observed data has to be adjusted so that the wind field is in balance with the pressure field with respect to the model. There are several methods of initialization, but in this lecture we will discuss only two of them, namely, static initialization and dynamic initialization.

#### 10.6.1 Static Initialization

In static initialization it is assumed that the divergence and the tendency of divergence is zero in the initial fields.

Consider the complete equation of horizontal divergence

$$\frac{\partial D}{\partial t} + \nabla \cdot D + D^2 - f\zeta + \beta t - 2J(u,v) + \nabla^2 \phi + \nabla \cdot \vec{F} = 0 \quad \text{…………10.62}$$

If we let $\frac{\partial D}{\partial t} = 0$ and $D(t=0) = 0$, and neglect the frictional term to then we have
\[ \nabla^2 \phi - f \zeta + \beta u - 2J(u,v) = 0 \]  
\[ \text{ Equation 10.63 } \]

Equation 10.62 is a non-linear balance equation. It represents the balance between the wind field and the geopotential field. If it is assumed that the wind field is accurate, then the initial geopotential field can be derived from the wind field using equation 10.63.

**Activity 10.6**

Show that equation 10.63 can be written as

\[ \nabla^2 \phi - f \nabla^2 \psi - \nabla \psi \cdot \nabla f + 2 \left( \frac{\partial^2 \psi}{\partial x \partial y} \right)^2 - 2 \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} = 0 \]

Or

\[ \nabla^2 \phi = \nabla \cdot (f \nabla \psi) + 2J \left( \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right) \]

Using the diagnostic equation

\[ \nabla^2 \phi = \nabla \cdot (f \nabla \psi) + 2J \left( \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y} \right) \]

We can use the following steps to obtain the geopotential

1. **Observed wind speed and direction**
2. **Resolve wind to u and v**
3. **Compute vorticity**
   \[ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \]
4. **Use relaxation method to find \( \phi \)**
5. **Output \( \phi \)**
6. **Use relaxation method to obtain Streamfunction from the equation**
   \[ \zeta = \nabla^2 \psi \]

**10.6.2 Dynamic Initialization**

In the dynamic initialization, time integration is performed backed and forward (using selective damping time Scheme) around the initial time to adjust wind and pressure fields until “HOPEFULLY” they are representative of the initial large scale meteorological variables.
Using Euler backward scheme in a simple equation such as
\[
\frac{\partial h}{\partial t} = F \quad \text{……………….. 10.65}
\]
Using a forward step we have
\[
h_{T+1} = h_T + \Delta t F_T \quad \text{……………….. 10.66a}
\]
\[
h_{T+1}^* = h_T + \Delta t F_{T+1}^* \quad \text{……………….. 10.66b}
\]
Equations 10.66a and 10.66b represent a one step forward using Euler scheme. The first step is needed to estimate the forcing \( F_{T+1}^* \)
This is followed by the back time step
\[
h^*_T = h_{T+1} - \Delta t F_{T+1}^* \quad \text{………………. 10.67a}
\]
\[
h_T = h_{T+1} - \Delta t F_T^* \quad \text{………………. 10.67b}
\]
The process given in equation 10.54 followed by 10.55 forms one complete loop.

The processes can also involve integrating a back time step from the initial time
Activity 10.7

Consider the non advective model

\[
\frac{\partial \mathbf{V}}{\partial t} = -\nabla \phi - \mathbf{f} \times \mathbf{V}
\]

\[
\frac{\partial \phi}{\partial t} = -H \nabla \cdot \mathbf{V}
\]

1. Write the first equation in the component form
2. Write down the finite difference equation that you would use to initialize the u,v and \( \phi \)

REFERENCES

Prandtl L 1952: Essentials of Fluid Dynamics. Blackie and Son limited
Hare F.K. 1961: The Restless Atmosphere. Hutchinson and Co. LTD
SAMPLE EXAM QUESTIONS- 1

Answer All Questions

Q1(a)(i) List the real forces that act on a parcel of air in the atmosphere. (1.5 marks)

(ii) State the two sources of centrifugal forces acting on a parcel of air in motion. (1 mark)

(iii) Explain why curvature may be neglected for large scale motions. (1.5 marks)

(iv) State the difference between natural coordinate and spherical coordinate. (1 mark)

(v) Although vertical accelerations do occur in the atmosphere, the atmosphere is largely in hydrostatic balance. Explain. (1.5 marks)

(vi) What is the difference between a prognostic and diagnostic equation? (1.5 marks)

(vii) State the advantage of representing the equation of motion in the (x, y, p, t) coordinate instead of the (x, y, z, t) coordinate (1 mark)

(viii) The geostrophic equation is not valid at the equator. Explain why? (1 mark)

(ix) Why is cyclostrophic flow rear around a high pressure. (1 mark)

(x) State the five parameters that may be used as vertical coordinate. (2 mark)

(b) The wind speed in m s\(^{-1}\) and direction at the points (I,J), (I + 1, J), (I, J-1), (I-1, J) and (I, J + 1) are (14, 340), (8, 190), (12, 280), (6, 350) and (10, 120) respectively. Compute the following at the point (I, J) if \(\Delta x = \Delta y = 200 km\).

i) Divergence (3 marks)
ii) Vorticity (3 marks)
iii) Deformation type I (3 marks)
iv) Deformation type II (3 marks)

Q2(a) Starting from the horizontal equation of motion

\[
\frac{du}{dt} = -\frac{\partial \phi}{\partial x} + fv + F \\
\frac{dv}{dt} = -\frac{\partial \phi}{\partial y} - fu + F
\]

derive the geostrophic equation stating the assumption (8 marks)

b) Using the equation you have derived in (a) show that the geostrophic vorticity is given by.

\[
\zeta_g = \frac{1}{f} \nabla^2 \phi + \frac{u_g}{\alpha} \text{Cot } \phi
\]

(8 marks)

c) Compute the geostrophic wind and geostrophic vorticity at the point (4°E, 45°N) where the geopotential height is 3020m, if the geopotential at the points (40°E, 47°N), (42°E, 45°N), (40°E, 42°N) and (38°E, 45°N) are 3060m, 3070m, 3040m and 3030m respectively. (9 marks)

Q3. A) Starting from the horizontal equation of motion.

\[
\frac{dv}{dt} = -\alpha \nabla p - f \vec{k} \times \vec{V} + \vec{F}
\]

derive the corresponding form in natural coordinate. (8 marks)

b) Starting the equation you have derived in (a) and making the necessary assumptions derive the equation of the gradient wind along a low in the southern hemisphere. (8 marks)

c) Determine the gradient wind speed in a cyclone at a point 800km from the centre at latitude 60°S if the isobar at 680km from the centre is 990mb and the isobar at 930 km from the centre is 998 mb. The density of the air is 1.27kgm⁻³. (9 marks)

SAMPLE EXAM QUESTION 2
Answer ALL Questions

Q1(a)(i) State the perturbation theory. (3 marks)

(ii) By considering the approximate vorticity equation

\[ \frac{d\eta}{dt} = \omega \]

where \( \eta = \zeta + f \)

Derive the equation for the phase speed of the Rossby wave (10 marks)

(b) Given

\[ \vec{v} = v_i - u_j \]

where \( x(t_o) = 3 \) and \( y(t_o) = -4 \)

(i) Derive the equation of the streamlines. (7 marks)

(ii) Find the position vector for the parcel of air at \( t = \pi/4 \) (10 marks)

Q2(a) The rotation of coordinates \( x, y \) to the new coordinates \( x_1, y_1 \), through an angle \( \theta \) is given by

\[ x_1 = x \cos \theta + y \sin \theta \]
\[ y_1 = x \sin \theta + y \cos \theta . \]

Show that relative vorticity is invariant with respect to rotation. (18 marks)

(b) Starting from \( \vec{v} = \frac{1}{2} D_x \hat{i} + \frac{1}{2} D_y \hat{j} \)

Show that the fractional time derivative of area is equal to Divergence. (12 marks)

Q3(a) Write down the equation relating streamfunction

(i) to rotation wind and divergence with to velocity potential. (4 marks)

(ii) starting from the observed wind field describe how you would obtain the divergent wind component. (10 marks)

(b) Given the horizontal wind in \( x, y, p, t \) coordinate as.
\[ \frac{d\vec{v}}{dt} = -\nabla \phi + f\hat{k} \times \vec{v} + \vec{F} \]

split this equation into i and j components and hence derive the vorticity equation (16 marks)

Q4(a) Starting from the hydrostatic equation show that the pressure tendency equation is

\[ \frac{\partial p}{\partial t} = -g \int z^a \nabla \cdot \rho \vec{v} dz + g \rho w \quad (15 \text{ marks}) \]

b) Starting from the three dimensional frictionless wind show that

\[ \frac{dc}{dt} = -\int_A (\nabla \alpha \cdot \nabla p) dA - 2\tilde{\Omega} \frac{dA}{dt} \quad (15 \text{ marks}) \]

Q5a)i) What is numerical weather prediction (NWP)? (2 marks)

(ii) Why is initialization important in NWP? (2 marks)

b) Differentiate between the following
i) implicit and explicit integration scheme (2 marks)
ii) computation mode and physical mode (2 marks)
iii) Backward and forward integration scheme (2 marks)
iv) Analytical solution and numerical solution (2 marks)
v) Truncation error and systematic error. (2 marks)
vi) Dynamic initialization and static initialization (2 marks)

c) Given a differential equation of the form

\[ \frac{\partial \zeta}{\partial t} + U \frac{\partial \zeta}{\partial x} = 0 \]

show that for computational stability

\[ \left| \frac{UVt}{Vx} \right| \leq 1 \quad (10 \text{ marks}) \]

d) \[ \frac{dq}{dt} = -kq \]

if \( k = 0.3 \) and \( q_0 = 2.5 \)

find \( q \) at \( t = 4 \) using:

(i) Analytical solution (3 marks)
(ii) Using numerical method with four time steps using forward integration scheme. (5 marks)